Solutions to Homework 7

FM 5021 Mathematical Theory Applied to Finance

10.2 Explain two ways in which a bear spread can be created.
A bear spread can be created using two call options with the same maturity and different strike prices. The investor shorts the call option with the lower strike price and buys the call option with the higher strike price.
A bear spread can also be created using two put options with the same maturity and different strike prices. The investor buys the put option with the higher strike price and shorts the put option with the lower strike price.

10.4 Call options on a stock are available with strike prices of $\$15$, $\$17\frac{1}{2}$, and $\$20$, and expiration dates in 3 months. Their prices are $\$4$, $\$2$, and $\$\frac{1}{2}$, respectively. Explain how the options can be used to create a butterfly spread. Construct a table showing how profit varies with stock price for the butterfly spread.
A butterfly spread can be created by buying call options with strike prices of \$15 and \$20 and shorting two call options with strike prices of $\$17\frac{1}{2}$. This requires an initial investment of $\$4 + \$\frac{1}{2} - 2 \times \$2 = \$\frac{1}{2}$. The table below shows the variation of the profit with the stock price at expiry.

Stock price, S_T	Profit
$S_T < 15$	$-\frac{1}{2}$
$15 < S_T < 17\frac{1}{2}$	$(S_T - 15) - \frac{1}{2}$
$17\frac{1}{2} < S_T < 20$	$(20 - S_T) - \frac{1}{2}$
$S_T > 20$	$-\frac{1}{2}$

10.8 Use put-call parity to relate the initial investment for a bull spread created using calls to the initial investment for a bull spread created using puts.
A bull spread using call options is created by buying a call option on a stock with a certain strike price and selling a call option on the same stock with a higher strike price and the same expiration date. A bull spread using put options is created by buying a put with a low strike price and selling a put with a higher strike price and same expiration date.
Define p_1 and c_1 to be the prices of a put and a call options with a strike price of K_1, and p_2 and c_2 to be the prices of a put and a call options with a strike price of K_2, where $K_1 < K_2$. From put-call parity we have that

$$p_1 + S = c_1 + K_1 e^{-rT}$$
$$p_2 + S = c_2 + K_2 e^{-rT}$$

where S is the current stock price, T is the time to maturity, and r is the risk-free interest rate. Thus,

$$p_1 - p_2 = c_1 - c_2 - (K_2 - K_1)e^{-rT}.$$

Since $(K_2 - K_1)e^{-rT} > 0$, the initial investment for a bull spread created using puts is less than the initial investment for a bull spread created using calls by an amount of $(K_2 - K_1)e^{-rT}$.
In fact, the initial investment when the bull spread is created using put options is negative, while the initial investment when the bull spread is created using call options is positive.

<u>10.11</u> Use put-call parity to show that the cost of a butterfly spread created from European puts is identical to the cost of a butterfly spread created from European calls.
Define c_1, c_2, and c_3 to be the prices of call options and p_1, p_2, and p_3 to be the prices of put options with strike prices of K_1, K_2, and K_3, respectively, where $K_1 < K_2 < K_3$ (K_2 is halfway between K_1 and K_2). A butterfly spread can be created by buying call options with strike prices of K_1 and K_3 and shorting 2 call options with strike prices of K_2. It can also be created by buying two put options with stock prices of K_1 and K_3 and shorting 2 put options with strike prices of K_2. With the notation from the previous problem, put-call parity gives

$$\begin{aligned} p_1 + S &= c_1 + K_1 e^{-rT} \\ p_2 + S &= c_2 + K_2 e^{-rT} \\ p_3 + S &= c_3 + K_3 e^{-rT}. \end{aligned}$$

Then,

$$p_1 + p_3 - 2p_2 = c_1 + c_3 - 2c_2 + (K_1 + K_3 - 2K_2)e^{-rT}.$$

Since $K_2 - K_1 = K_3 - K_2 \implies K_1 + K_3 - 2K_2 = 0$, we conclude that

$$c_1 + c_3 - 2c_2 = p_1 + p_3 - 2p_2,$$

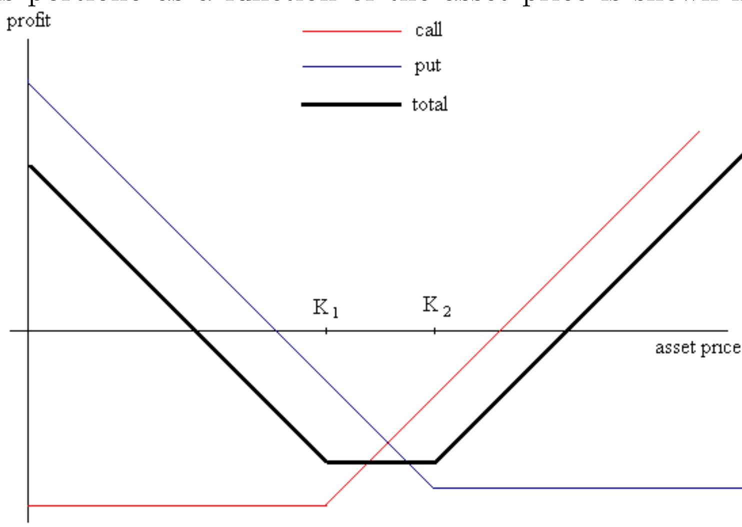
i.e. the cost of a butterfly spread created from European puts is identical to the cost of a butterfly spread created from European calls.

<u>10.15</u> How can a forward contract on a stock with a particular delivery price and delivery date be created from options?
Let K be the delivery price and T be the delivery date. A forward contract is created by buying a European call and selling a European put when both options have strike prices of K and expiry dates T. This portfolio has a payoff of $S_T - K$ under all circumstances, where S_T is the stock price at time T. If $K = F_0$, the forward contract that is created has zero value. This shows that the price of a call equals the price of a put when the strike price is F_0.

<u>10.17</u> What is the result if the strike price of the put is higher than the strike price of the call in a strangle?
In a strangle an investor buys a put and a call with the same expiry date and different

Range of stock price	Payoff from call	Payoff from put	Total payoff
$S_T \leq K_1$	0	$K_2 - S_T$	$K_2 - S_T$
$K_1 < S_T < K_2$	$S_T - K_1$	$K_2 - S_T$	$K_2 - K_1$
$S_T \geq K_2$	$S_T - K_1$	0	$S_T - K_1$

strike prices. Let K_1 be the strike price of the call and K_2 be the strike price of the put. The table above gives the variation of payoff of the strangle with the asset price at expiry. The profit of this portfolio as a function of the asset price is shown in the figure below.



11.2 Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

In the no-arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on the portfolio equal to the risk-free interest rate, we are able to value the option.

In the risk-neutral valuation approach, we first choose probabilities for the branches of the tree so that the expected return on the stock equals the risk-free interest rate. We then value the option by calculating its expected payoff and discounting this expected payoff at the risk-free interest rate.

11.3 What is meant by the "delta" of a stock option?

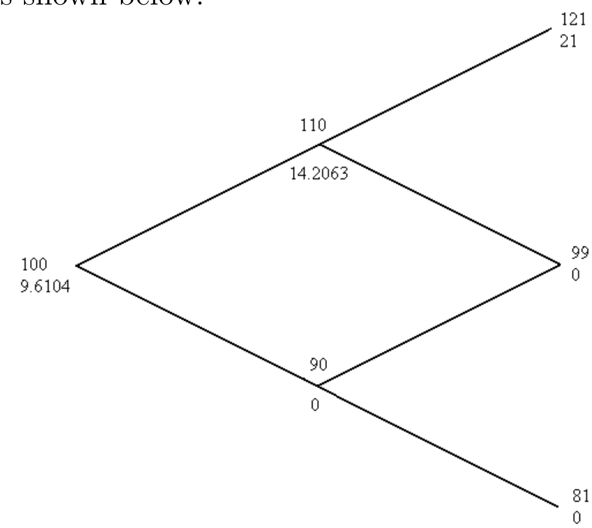
The "delta" of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying asset. It measures the sensitivity of the option price to the price of the stock when small changes are considered.

11.5 A stock price is currently $\$100$. Over each of the next two 6-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-year European call option with a strike price of $\$100$?

We have that $u = 1.10$, $d = 0.90$, $\Delta t = 0.5$, and $r = 0.08$. Then, the risk-neutral probability of an up move is given by

$$p = \frac{e^{0.08\times 0.5} - 0.90}{1.0 - 0.90} = 0.7041.$$

We can work back from the end of the tree to the beginning to find the value of the option to be \$9.61. The tree is shown below.

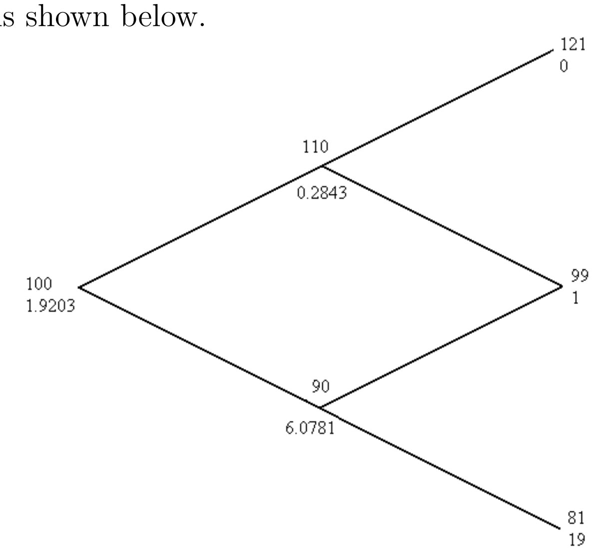


We can also use equation (11.10) directly to find the option value

$$\left[0.7041^2 \times \$21 + 2\times 0.7041 \times (1-0.7041)\times \$0 + (1-0.7041)^2 \times \$0\right] e^{-2\times 0.08\times 0.5} = \$9.61.$$

11.6 For the situation considered in Problem 11.5, what is the value of a 1-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put-call parity.

We can work back from the end of the tree to the beginning to find the value of the option to be \$1.92. The tree is shown below.



The option value can be calculated directly from equation (11.10)

$$\left[0.7041^2 \times \$0 + 2\times 0.7041 \times (1-0.7041)\times \$1 + (1-0.7041)^2 \times \$19\right] e^{-2\times 0.08\times 0.5} = \$1.92.$$

The stock price plus the put price is $\$100 + \$1.92 = \$101.92$. The present value of the strike price plus the call price is $\$100e^{-0.08\times 1} + \$9.61 = \$101.92$, verifying that the put-call parity holds.

11.7 What are the formulas for u and d in terms of volatility?

$$u = e^{\sigma\sqrt{\Delta t}} \text{ and } d = e^{-\sigma\sqrt{\Delta t}}$$

11.8 Consider the situation in which stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

The riskless portfolio consists of a short position in the option and a long position in Δ shares. Because Δ changes during the life of the option, the riskless portfolio must also change.

11.12 A stock price is currently $\$50$. Over each of the next two 3-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 6-month European call option with a strike price of $\$51$?

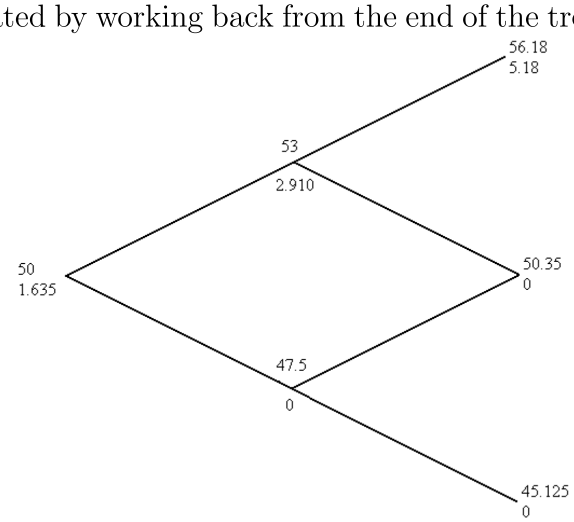
The price of the stock in case of two up moves would be $\$50 \times 1.06^2 = \$56.18$; in case of two down moves, it would be $\$50 \times 0.95^2 = \$45.125$, and in case of one up move and one down move it would be $\$50 \times 1.06 \times 0.95 = \$45.125$. Because the strike price is $\$51$, the payoff in case of two up moves is $\$56.18 - \$50 = \$5.18$ and zero in all other cases.
The risk-neutral probability of an up move is given by

$$p = \frac{e^{0.05\times\frac{3}{12}} - 0.95}{1.06 - 0.95} = 0.5689.$$

The value of the option is, therefore,

$$\$5.18 \times 0.5689^2 \times e^{-0.05\times\frac{6}{12}} = \$1.635.$$

This can also be calculated by working back from the end of the tree, which is shown below.

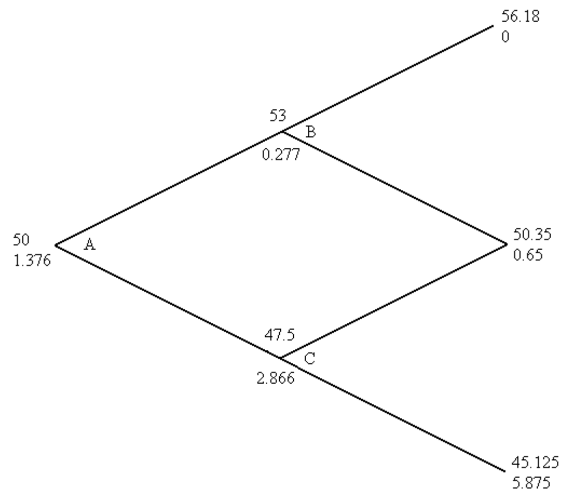


__11.13__ **For the situtation considered in Problem 11.12, what is the value of a 6-month European put option with a strike price of $51? Verify that the European call and European put prices satisfy put-call parity. If the put option were American, would it ever be optimal to exercise it early at any of the nodes of the tree?**

We get a payoff of $\$51 - \$50.35 = \$0.65$ if the middle final node is reached (one up and one down move) and a payoff of $\$51 - \$45.125 = \$5.875$ if the lowest final node is reached (two down moves). The value of the option is, therefore,

$$\left[\$0.65 \times 2 \times 0.5689 \times (1 - 0.5689) + \$5.875 \times (1 - 0.5689)^2\right] e^{-0.05\times\frac{6}{12}} = \$1.376.$$

This can alse be calculated if we work back from the end of the tree, which is shown below.



The value of the put plus the stock price (from 11.12) is

$$\$1.376 + \$50 = \$51.376.$$

The value of the call plus the present value of the strike price is

$$\$1.635 + \$51e^{-0.05\times\frac{6}{12}} = \$51.376,$$

so that the put-call parity holds.

To test whether it is worth exercising the option early we compare the value calculated for the option at each node with the payoff from immediate exercise. At node C the payoff from immediate exercise is $\$51 - \$47.5 = \$3.5$. Because this is greater than $2.866, the option **should** be exercised at node C. The option should not be exercised at either node A or node B.

__11.17__ **A stock price is currently $40. Over each of the next two 3-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.**

(a) What is the value of a 6-month European put option with a strike price of $42?

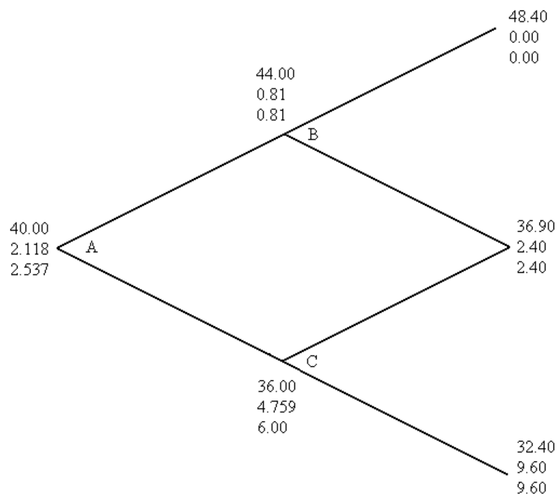
The risk-neutral probability of an up move is

$$p = \frac{e^{0.12\times\frac{3}{12}} - 0.90}{1.10 - 0.90} = 0.6523.$$

The price of the stock in case of two up moves would be $\$40 \times 1.10^2 = \$48.40$; in case of one up and one down move it would be $\$40 \times 1.10 \times 0.90 = \$39.60$, and in case of two down moves it would be \$32.40. The payoffs are \$0.00, \$2.40, and \$9.60, respectively. Therefore, the value of the option is

$$\left[\$2.40 \times 2 \times 0.6523 \times (1 - 0.6523) + \$9.60 \times (1 - 0.6523)^2\right] e^{-0.12\times\frac{6}{12}} = \$2.188.$$

This could also be calculated by working back from the end of the tree, which is shown below. The second number at each node is the value of the European option.



(b) What is the value of a 6-month American put option with a strike price of $42?

The value of the American option is shown in the tree as the third number at each node. It is \$2.537. Note that it is greater than the value of the European put option because it is optimal to exercise early at node C.