Chapter 12

Analysis of Variance

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- Know how 'analysis of variance' (ANOVA) can be used to test for the equality of three or more population means.

- the terms like 'response variable', 'a factor' and 'a treatment', and use them in the analysis of variance.

- Know how to summarize F-ratio in the form of an ANOVA table.

12.1   INTRODUCTION

In Chapter 10, we introduced procedures to test the significance of difference between two sample means to understand whether the means of two populations are equal based upon two independent random samples. In such all cases, the null hypothesis states that there is no significant difference between population mean, i.e., $H_0 : \mu_1 = \mu_2$. However, there may be situations where more than two populations are involved and we need to test the significance of difference among their sample means. This implies that a null hypothesis be stated that more than two populations from which independent samples are drawn have equal mean values against the alternative hypothesis that population mean values are not all equal. Let $\mu_1, \mu_2, ..., \mu_r$ be the mean value for population $1, 2, ..., r$, respectively. Then from sample data we intend to test the following hypotheses:

$$H_0 : \mu = \mu_2 = ... = \mu_r$$

and

$$H_1 : \text{Not all } \mu_j \text{ are equal } (j = 1, 2, ..., r)$$

In other words, the null hypothesis should be rejected if any one of the r sample means is different from the others.

For example, the production level in three shifts in a factory needs to be compared to answer question such as `is the production level higher/lower on any day of the week?'. Moreover, production level on any day and shift of the week in different combinations can also be analysed.

Few examples of more than two populations where it is necessary to conduct a comparative study to arrive at a statistical inference are as follows:

- Effectiveness of different promotional activities in term of sales.
- Quality of a product produced by different manufacturers in terms of an attribute.
- Yield of crop due to varieties of seeds, fertilizers and quality of soil.

Analysis of Variance: A statistical procedure for determining whether the means of several different populations are equal.

Under certain circumstances, it is not desirable to conduct repeated t-tests on pairs of the samples because it reduces the probability of the outcome being correct for the combined results. To overcome this problem, a method known as *analysis of variance* (*ANOVA*) developed by R. A. Fisher is used to test the significance of the difference between more than two population means.

Terms To Be Used for Analysis of Variance

- *Sampling plan or experimental design* is the procedure to select a sample from the population under study and determines the value of desired sample statistic.
- *Experimental unit* is the object or an element on which a measurement or measurements is taken. The conditions imposed on an experimental unit provide effect on the response.
- *Factor or criterion* is an independent variable whose values are controlled by the researcher.
- *Level* is the intensity for setting of a factor.
- *Treatment or population* is a specific combination of factor levels.
- *Response* is the dependent variable being measured.

Factor: Another word for independent variable of interest that is controlled in the analysis of variance.

For example, a tyre manufacturing company plans to conduct a tyre-quality study in which quality is the independent variable called *factor or criterion* and the *treatment levels or classifications* are low, medium and high quality. The *dependent* (*or response*) variable might be the number of kilometers driven before tyre is rejected for use. A study of daily sales volumes at various zones may be taken as the independent variable. A treatment levels or classifications would location of stores inner-city, metro-cities, state capitals and small towns. The dependent variable would be sales in rupees.

12.2 ANALYSIS OF VARIANCE APPROACH

The analysis of variance starts with partitioning or dividing the total variation in the sample data into the following two components in such a way that it is possible to estimate the contribution of factors that may cause variation.

(i) The amount of variation *among the sample means*. This variation is either due to difference in treatment or due to element of chance. This difference is denoted by SSC or SSTR.

(ii) The amount of variation *within the sample observations*. This difference may be due to chance or experimental (random) errors. The difference among value of observations in a sample due to chance is called an estimate and denoted by SSE.

Factor Level: A value at which the factor is controlled.

The observations in the sample may be classified according to *one factor* (criterion) or *two factors* (criteria). The classification according to one factor and two factors is called *one-way classification* and *two-way classification*, respectively. The calculations for total variation may be carried out in each of the two types of classifications by

- Direct method
- Short-cut method
- Coding method

Assumptions for Analysis of Variance

- Each population under study is normally distributed with a mean μ_r that may or may not be equal but with equal variances, σ^2_r.
- Each sample is drawn randomly and independent of other samples.

12.3   TESTING EQUALITY OF POPULATION MEANS: ONE-WAY CLASSIFICATION

One-way Analysis of Variance: Analysis of variance in which only one criterion (variable) is used to analyse the difference between more than two population means.

In many real-life problems, observations in a data set are classified into several groups based on a single attribute of interest, such a (i) *percentage of marks* secured by students in a paper, (ii) *flavour preference* of ice-cream by customers, and (iii) *yield of crop* due to varieties of seeds and so on. Such type of classification of observations in the data set into several groups based on a single attribute is called *one-way classification.*

For all theoretical purposes, populations (several groups classified based on single factor or criterion in a data set) are referred as treatments. The effect of a factor (criterion such as flavour preference) on the dependent variable (sale) on different groups (variety of ice-creams) is required to be studied. These groups are the treatments in this example.

Suppose inferences are to be made about r population means μ_1, μ_2, ..., μ_r based on independent random samples of size n_1, n_2, ..., n_r from normal populations with a common variances σ^2. The null hypothesis to be tested is stated as

$$H_0 : \mu_1 = \mu_2 = ... = \mu_r \qquad \leftarrow \text{Null hypothesis}$$
$$H_1 : \text{Not all } \mu_j \,(j = 1, 2, ..., r) \text{ are equal} \quad \leftarrow \text{Alternative hypothesis}$$

Let n_j = size of the jth sample ($j = 1, 2, ..., r$)

n = number of observations in all samples combined (i.e. $n = n_1 + n_2 + ... + n_r$)

x_{ij} = value of ith observation within the sample from jth population

The value of observations obtained for r independent samples based on one-criterion classification can be arranged as shown in Table 12.1.

Table 12.1   One-criterion Classification of Data

Observations	Populations (Number of Samples)			
	1	2	...	r
1	x_{11}	x_{12}	...	x_{1r}
2	x_{21}	x_{22}	...	x_{2r}
⋮	⋮			
k	x_{k1}	x_{k2}	...	x_{kr}
Sum	T_1	T_2	...	$T_r = T$
A.M.	$\bar{x}_1$	$\bar{x}_2$	...	$\bar{x}_r = \bar{\bar{x}}$

where $$T_i = \sum_{i=1}^{k} x_{i1} \qquad\qquad T = \sum_{j=1}^{r} T_i$$

$$\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_{i1} \qquad \text{Grand mean, } \bar{\bar{x}} = \frac{1}{rk}\sum_{j=1}^{r} \bar{x}_j = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{r} x_{ij}$$

The values of $\bar{x}_i$ are called sample means and $\bar{\bar{x}}$ are the grand mean of all observations (or measurements) in all the samples.

Since there are k rows and r columns in Table 12.1, therefore total number of observations are $rk = n$, provided each row has equal number of observations. But if the number of observations in each row varies, then the total number of observations are $n_1 + n_2 + ... + n_r = n$.

Illustration Three brands A, B and C of tyres were tested for durability. A sample of four tyres of each brand is subjected to the same test and the number of km until wear out was noted for each brand of tyres. The data in thousand km is given in Table 12.2.

Table 12.2   Example of Data in ANOVA

Observations	Population (Number of Brands)		
	A	B	C
1	26	18	23
2	25	16	19
3	28	17	26
4	12	18	30
Sum	91	69	98
Sample size	4	4	4
Mean	22.75	17.25	24.50

Since the same number of observations are obtained for each brand of tyres (population), therefore the number of observations in the table are $n = rk = 3 \times 4 = 12$.

- The sample mean of each of three samples is given by

$$\bar{x}_1 = \frac{1}{4} \sum_{i=1}^{4} x_{i1} = \frac{1}{4}(91) = 22.75$$

$$\bar{x}_2 = \frac{1}{4} \sum_{i=1}^{4} x_{i2} = \frac{1}{4}(69) = 17.25 \text{ and } \bar{x}_3 = 24.50$$

- The grand mean for all samples is

$$\bar{\bar{x}} = \frac{1}{3}(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)$$

$$= \frac{1}{3}(22.75 + 17.25 + 24.50) = 21.50$$

12.3.1   Steps for Testing Null Hypothesis

Step 1 : State the null and alternative hypotheses to test the equality of population means as

$$H_0 : \mu_1 = \mu_2 = ... = \mu_r \qquad \leftarrow \text{Null hypothesis}$$
$$H_1 : \text{Not all } \mu_j\text{s are equal } (j = 1, 2, ..., r) \leftarrow \text{Alternative hypothesis}$$

α = level of significance

Step 2 : Calculate Total Variation If single sample of size n is taken from the population, then estimation of the population variance based on the variance of sampling distribution of mean is given by

$$s^2 = \frac{\sum(x - \bar{x})^2}{n - 1} = \frac{SS}{df}$$

The numerator in s^2 is called *sum of squares* of deviations of sample values about the sample mean $\bar{x}$ and is denoted as SS. Consequently, 'sum of squares' is a measure of variation. Thus, when SS is divided by $df = n - 1$, the result is often called the *mean square* which is an alternative term for **sample variance**.

Total variation is represented by the '**sum of squares total**' (**SST**) and is equal to the sum of the squared differences between each sample value from the grand mean $\bar{\bar{x}}$

$$\text{SST} = \sum_{i=1}^{r} \sum_{j=1}^{n_j} (x_{ij} - \overline{\overline{x}})^2$$

where r is number of samples (or treatment levels) and n_j is size of jth sample.

The total variation is divided into two parts as shown below:

Total variation	Variation between (or among) sample means (Also called *sum of squares for treatment*)
	Variation within the samples values (Also called *sum of squares for error*)
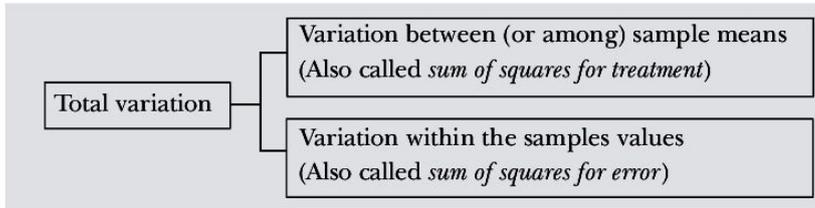
Step 3 : Calculate Variation Between Sample Means This is usually called the 'sum of squares between' and measures the variation between samples due to treatments. In statistical terms, variation between samples means is also called the *between-column variance*. The procedure is as follows:

(a) Calculate mean values: $\overline{x}_1, \overline{x}_2, \dots \overline{x}_r$ of all r samples

(b) Calculate grand mean $\overline{\overline{x}} = \dfrac{1}{r} (\overline{x}_1 + \overline{x}_2 + \dots + \overline{x}_r) = \dfrac{\text{T}}{n}$

where T is the grand total of all observations and n is the number of observations in all r samples.

(c) Calculate difference between the mean of each sample and the grand mean as $\overline{x}_1 - \overline{\overline{x}}, \overline{x}_2 - \overline{\overline{x}}, \dots, \overline{x}_r - \overline{\overline{x}}$. Multiply each of these by the number of observations in the corresponding sample and add. The total gives the sum of the squared differences between the sample means in each group and is denoted by SSC or SSTR.

$$\text{SSTR} = \sum_{j=1}^{r} n_j(\overline{x}_j - \overline{\overline{x}})^2$$

This sum is also called *sum of squares for treatment* **(SSTR).**

Step 4 : Calculate Variation Within Samples This is usually called the 'sum of squares within' and measures the difference within samples due to chance error. Such variation is also called *within sample variance*. The procedure is as follows:

(a) Calculate mean values $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_r$ of all r samples.
(b) Calculate difference of each observation in r samples from the mean values of the respective samples.
(c) Square all the differences obtained in Step (b) and find the total of these differences.
The total gives the sum of the squares of differences within the samples and is denoted by SSE.

$$\text{SSE} = \sum_{i=1}^{k} \sum_{j=1}^{n_j} (x_{ij} - \overline{x}_j)^2$$

This sum is also called the *sum of squares for error*, **SSE = SST – SSTR.**

Step 5 : Calculate Average Variation Between and Within Samples—Mean Squares Since r independent samples are being compared, therefore $r - 1$ degrees of freedom are associated with the sum of the squares among samples. As each of the r samples contributes $n_j - 1$ degrees of freedom for each independent sample within itself, therefore there are $n - r$ degrees of freedom associated with the sum of the squares within samples. Thus, total degrees of freedom equal to the degrees of freedom associated with SSC (or SSTR) and SSE. That is

Total df = Between samples (treatments) df + Within samples (error) df
$$n - 1 = (r - 1) + (n - r)$$

Divide 'sum of squares' by their associated degrees of freedom to get *variances or mean square* terms as follows:

$$MSTR = \frac{SSTR}{r - 1} \; ; MSE = \frac{SSE}{n - r} ;$$

and

$$MST = \frac{SST}{n - 1}$$

It may be noted that the quantity $MSE = SSE/(n - r)$ is a pooled estimate of population variance, σ^2 (weighted average of all r sample variances whether H_0 is true or not)

Step 6 : Apply F-test statistic with $r - 1$ degrees of freedom for the numerator and $n - r$ degrees of freedom for the denominator

$$F = \frac{\sigma^2_{between}}{\sigma^2_{within}}$$

$$= \frac{SSTR/(r - 1)}{SSE/(n - r)} = \frac{MSTR}{MSE}$$

Step 7 : Make Decision Regarding Null Hypothesis If the calculated value of F-test statistic is more than its right tail critical value $F_{(r - 1, n - r)}$ at a given level of significance α and degrees of freedom $r - 1$ and $n - r$, then reject the null hypothesis. In other words, as shown in Fig. 12.1, the decision rule is

- If the calculated value of F > its critical value $F_{\alpha(r - 1, n - r)}$, reject null hypothesis, H_0
- Otherwise accept H_0

Remarks: In F-distribution table, columns represent the degrees of freedom for numerator and rows represent the degrees of freedom for denominator.

Figure 12.1
Rejection Region for Null Hypothesis using ANOVA
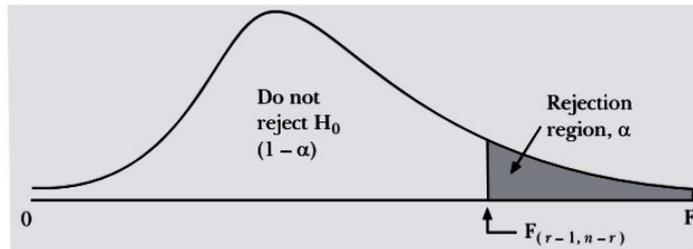
If null hypothesis H_0 is true, then the variance in the sample means measured by $MSTR = SSTR/(r - 1)$ provides an unbiased estimate of population variance, σ^2. But if H_0 is false and population means are different, then MSTR is as large as shown in Fig 12.2

Figure 12.2
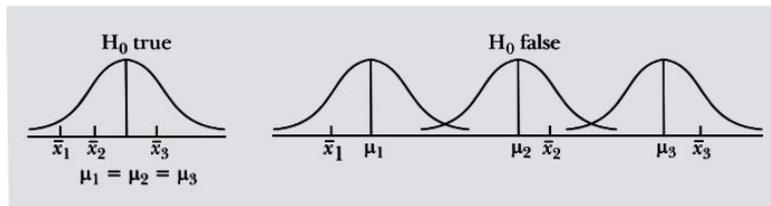Sample Means Drawn from Identical Populations

Table 12.3 shows the general arrangement of the ANOVA table for one-factor analysis of variance.

Table 12.3 ANOVA Summary Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	Test-Statistic or F-Value
• Between samples (Treatments)	SSTR	$r - 1$	$MSTR = \dfrac{SSTR}{r-1}$	
				$F = \dfrac{MSTR}{MSE}$
• Within samples error	SSE	$n - r$	$MSE = \dfrac{SSE}{n-r}$	
Total	SST	$n - 1$		

ANOVA Table: A standard table used to summarize the analysis of variance calculations and results.

Short-cut Method

The values of SSTR and SSE can be calculated as follows:
- Calculate the grand total of all observations in samples, T

$$T = \sum x_1 + \sum x_2 + ... + \sum x_r$$

- Calculate the correction factor $CF = \dfrac{T^2}{n}; n = n_1 + n_2 + ... + n_r$

- Find the sum of the squares of all observations in each of r samples and subtract CF form this sum to obtain the ***total sum of the squares of deviations, SST:***

$$SST = (\sum x_1^2 + \sum x_2^2 + ... + \sum x_r^2) - CF = \sum_{i=1}^{k} \sum_{j=1}^{n_j} x_{ij}^2 - CF$$

$$SSTR = \frac{(\sum x_j)^2}{n_j} - CF$$

and $\qquad SSE = SST - SSTR$

Coding Method

In the above method, computational time increases when numerical values of observations are in more than one digit. Since F-test statistic is the ratio of variances without unit of measurement, its values does not change when an appropriate constant value is either multiplied, divided, subtracted or added to each of the observations in the sample data. This adjustment reduces the digits of numerical values in the sample data and hence reduces computational time to calculate F value without any change.

Example 12.1: To test the significance of variation in the retail prices of a commodity in three metro cities, Mumbai, Kolkata and Delhi, four shops were chosen at random in each city and the prices observed in rupees were as follows:

Mumbai :	16	8	12	14
Kolkata :	14	10	10	6
Delhi :	4	10	8	8

Do the data indicate that the prices in the three cities are significantly different?

[Jammu Univ., M.Com, 2005]

Solution: Let us take the null hypothesis that there is no significant difference in the prices of a commodity in the three cities. Calculations for analysis of variance are as under.

Sample 1		Sample 2		Sample 3	
Mumbai		Kolkata		Delhi	
x_1	x_1^2	x_2	x_2^2	x_3	x_3^2
16	256	14	196	4	16
8	64	10	100	10	100
12	144	10	100	8	64
14	196	6	36	8	64
50	660	40	432	30	244

There are $r=3$ treatments (samples) with $n_1=4$, $n_2=4$, $n_3=4$ and $n=12$.

$$T = \text{Sum of all the observations in the three samples}$$
$$= \Sigma x_1 + \Sigma x_2 + \Sigma x_3 = 50 + 40 + 30 = 120$$

$$CF = \text{Correction factor} = \frac{T^2}{n} = \frac{(120)^2}{12} = 1200$$

$$SST = \text{Total sum of the squares}$$
$$= (\Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2) - CF$$
$$= (660 + 432 + 244) - 1200 = 136$$

$$SSTR = \text{Sum of squares between the samples}$$

$$= \left(\frac{(\Sigma x_1)^2}{n_1} + \frac{(\Sigma x_2)^2}{n_2} + \frac{(\Sigma x_3)^2}{n_3} \right) - CF$$

$$= \left\{ \frac{(50)^2}{4} + \frac{(40)^2}{4} + \frac{(30)^2}{4} \right\} - 1200$$

$$= \left\{ \frac{2500}{4} + \frac{1600}{4} + \frac{900}{4} \right\} - 1200$$

$$= \frac{5000}{4} - 1200 = 50$$

$$SSE = SST - SSTR = 136 - 50 = 86$$

Degrees of freedom: $df_1 = r - 1 = 3 - 1 = 2$ and $df_2 = n - r = 12 - 3 = 9$

Thus, $MSTR = \dfrac{SSTR}{df_1} = \dfrac{50}{2} = 25$ and $MSE = \dfrac{SSE}{df_2} = \dfrac{86}{9} = 9.55$

Table 12.4 ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	Test-Statistic
• Between samples	50	2	25	$F = \dfrac{25}{9.55}$
• Within samples	86	9	9.55	$= 2.617$
Total	136	11		

Since calculated value of F is less than its critical value, 4.26 at $df_1 = 2$, $df_2 = 9$, and $\alpha = 5$ per cent level of significance, the null hypothesis is accepted. Hence, we conclude that the prices of a commodity in three cities have no significant difference.

Example 12.2: A study was conducted to investigate the perception of corporate ethical values among individuals specializing in marketing. Using $\alpha = 0.05$ and the following data (higher scores indicate higher ethical values), test for significant differences in perception among three groups.

Marketing Manager	Marketing Research	Advertising
6	5	6
5	5	7
4	4	6
5	4	5
6	5	6
4	4	6

Solution: Let us take the null hypothesis that there is no significant difference in ethical values among individuals specializing in marketing. Calculations for analysis of variance are as under:

Marketing Manager		Marketing Research		Advertising	
x_1	x_1^2	x_2	x_2^2	x_3	x_3^2
6	36	5	25	6	36
5	25	5	25	7	49
4	16	4	16	6	36
5	25	4	16	5	25
6	36	5	25	6	36
4	16	4	16	6	36
30	154	27	123	36	218

There are $r = 3$ treatments (samples) with $n_1 = n_2 = n_3 = 6$ and $n = 18$.

$$T = \text{Sum of all the observations in three samples}$$
$$= \Sigma x_1 + \Sigma x_2 + \Sigma x_3 = 30 + 27 + 36 = 93$$

$$CF = \text{Correction factor} = \frac{T^2}{n} = \frac{(93)^2}{18} = 480.50$$

$$SST = \text{Total sum of the squares}$$
$$= (\Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^{2)} - CF$$
$$= (154 + 123 + 218) - 480.50 = 14.50$$

$$SSTR = \text{Sum of squares between the samples}$$

$$= \left(\frac{(\Sigma x_1)^2}{n_1} + \frac{(\Sigma x_2)^2}{n_2} + \frac{(\Sigma x_3)^2}{n_3} \right) - CF$$

$$= \left\{ \frac{(30)^2}{6} + \frac{(27)^2}{6} + \frac{(36)^2}{6} \right\} - 480.50$$

$$= \left(\frac{900}{6} + \frac{729}{6} + \frac{1296}{6} \right) - 480.50$$

$$= (150 + 121.5 + 216) - 480.50 = 7$$

$$SSE = SST - SSTR = 14.50 - 7 = 7.50$$

Degrees of freedom: $df_1 = r - 1 = 3 - 1 = 2$ and $df_2 = n - r = 18 - 3 = 15$

Thus, $\quad MSTR = \dfrac{SSTR}{df_1} = \dfrac{7}{2} = 3.5$ and $MSE = \dfrac{SSE}{df_2} = \dfrac{7.50}{15} = 0.5$

Table 12.5 ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	Test-Statistic
• Between samples	7	2	3.5	F = 3.5/0.5
• Within samples	7.5	15	0.5	= 7
Total	14.5	17		

Since calculated value of F = 7 is more than its critical value, F = 3.68 at df_1 = 2, df_2 = 15 and α = 0.05, the null hypothesis is rejected. Hence, we conclude that there is significant difference in ethical values among individuals specializing in marketing.

Example 12.3: As head of the department of a consumer's research organization, you have the responsibility for testing and comparing lifetimes of four brands of electric bulbs. Suppose you test the lifetime of three electric bulbs of each of the four brands. The data are shown below, each entry represents the lifetime (in hundreds of hours) of an electric bulb.

	Brand		
A	B	C	D
20	25	24	23
19	23	20	20
21	21	22	20

Can we infer that the mean lifetimes of the four brands of electric bulbs are equal?

[*Roorkee Univ., MBA, 2002*]

Solution: Let us take the null hypothesis that the mean lifetime of the four brands of electric bulbs is equal.

Subtract a common figure 20 from each observation. The calculations with coded data are as under:

Sample 1 (A)		Sample 2 (B)		Sample 3 (C)		Sample 4 (D)	
x_1	x_1^2	x_2	x_2^2	x_3	x_3^2	x_4	x_4^2
0	0	5	25	4	16	3	9
–1	1	3	9	0	0	0	0
1	1	1	1	2	4	0	0
0	2	9	35	6	20	3	9

$$T = \text{Sum of all the observations in four samples}$$

$$= \Sigma x_1 + \Sigma x_2 + \Sigma x_3 + \Sigma x_4 = 0 + 9 + 6 + 3 = 18$$

$$CF = \text{Correction factor} = \frac{T^2}{n} = \frac{(18)^2}{12} = 27$$

$$SST = \text{Total sum of the squares}$$

$$= (\Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2) - CF$$

$$= (2 + 35 + 20 + 9) - 27 = 39$$

$$SSTR = \text{Sum of squares between the samples}$$

$$= \left\{ \frac{(\Sigma x_1)^2}{n_1} + \frac{(\Sigma x_2)^2}{n_2} + \frac{(\Sigma x_3)^2}{n_3} + \frac{(\Sigma x_4)^2}{n_4} \right\} - CF$$

$$= \left\{ 0 + \frac{(9)^2}{3} + \frac{(6)^2}{3} + \frac{(3)^2}{3} \right\} - 18$$

$$= (0 + 27 + 12 + 3) - 27 = 15$$

$$SSE = SST - SSTR = 39 - 15 = 24$$

Degrees of freedom: $df_1 = r - 1 = 4 - 1 = 3$, $df_2 = n - r = 12 - 4 = 8$. Thus

$$\text{MSTR} = \frac{\text{SSTR}}{df_1} = \frac{15}{3} \text{ and MSE} = \frac{\text{SSE}}{df_2} = \frac{24}{8} = 3$$

Table 12.6 ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	Test Statistic
• Between samples	15	3	5	F = 5/3 = 1.67
• Within samples	24	8	3	
Total	39	11		

Since the calculated value of $F = 1.67$ is less than its critical value, $F = 4.07$ at $df_1 = 3$, $df_2 = 8$, and $\alpha = 0.05$, the null hypothesis is accepted. Hence, we conclude that the difference in the mean lifetime of four brands of bulbs is not significant.

12.4 INFERENCES ABOUT POPULATION MEANS

When null hypothesis, H_0 is rejected, it implies that all population means are not equal. However, we may not be satisfied with this conclusion and may want to know which population means differ. The answer to this question comes from the construction of confidence intervals using the small sample procedures based on t-distribution.

For a single population mean, μ, the confidence interval is given by

$$\bar{x} \pm t_{\alpha/2}\left(s/\sqrt{n}\right)$$

where $\bar{x}$ is the sample mean from a population. Similarly, confidence interval for the difference between two population means μ_1 and μ_2 is given by

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

where $\bar{x}_1$ and $\bar{x}_2$ are sample means from population 1 and 2, respectively, and n_1 and n_2 are number of observations in sample 1 and 2, respectively.

However, the following information is required to use confidence intervals:
- Value of s or s^2 as the best estimate of the common sample variance?
- Degrees of freedom used for the critical value of t-test statistic?

Regardless of test or estimation used, $s^2 = \text{MSE}$ or $s = \sqrt{\text{MSE}} = \sqrt{\text{SSE}/(n - r)}$ with $df = (n - r)$ and $t_{\alpha/2}$ at specified level of significance α is used to estimate σ^2, where $n = n_1 + n_2 + \dots + n_r$, and MSE $= \text{SSE}/(n - r)$ *called mean square error.*

Illustration Refer Example 12.1 where $\bar{x}_1 = 5$; $\bar{x}_3 = 6$, $n = n_1 + n_2 + n_3 = 18$, $s^2 = MSE = 0.5$ and $\alpha = 0.05$ level of significance. So, the confidence interval is computed as

$$(\bar{x}_1 - \bar{x}_3) \pm t_{\alpha/2}\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = (5-6) \pm 2.131\sqrt{0.5\left(\frac{1}{6} + \frac{1}{6}\right)}$$

$$= -1 \pm 1.225 = -2.225 \text{ and } 0.225$$

Since zero is included in this interval, we may conclude that there is no significant difference in the selected population means. That is, there is no difference between ethical values of marketing and advertising managers.

Remark: If end points of the confidence interval have the same sign, then it may be concluded that there is a significant difference between the selected population means.

Self-practice Problems 12A

12.1 Kerala Traders Co. Ltd wishes to test whether its three salesmen A, B and C tend to make sales of the same size or whether they differ in their selling ability as measured by the average size of their sales. During the last week there have been 14 sales calls—A made 5 calls, B made 4 calls, and C made 5 calls. Following are the weekly sales record of the three salesmen:

A	:	300	400	300	500	0
B	:	600	300	300	400	—
C	:	700	300	400	600	500

Perform the analysis of variance and draw your conclusions.

[*Madras Univ., M.Com., 2004; Madurai Univ., M.Com., 2004*]

12.2 There are three main brands of a certain powder. A sample of 120 packets sold is examined and found to be allocated among four groups A, B, C and D, and brands I, II and III as shown below:

Brand	Group			
	A	B	C	D
I	0	4	8	15
II	5	8	13	6
III	18	19	11	13

Is there any significant difference in brand preferences?

12.3 An agriculture research organization wants to study the effect of four types of fertilizers on the yield of a crop. It divided the entire field into 24 plots of land and used fertilizer at random in 6 plots of land. Parts of the calculations are shown below:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F-Test Statistic
• Fertilizers	2940	3	—	
				5.99
• Within groups	—	—	—	
Total	6212	24		

(a) Fill in the blanks in the ANOVA table.
(b) Test at $\alpha = 0.05$, whether the fertilizers differ significantly.

12.4 A manufacturing company has purchased three new machines of different makes and wishes to determine whether one of them is faster than the others in producing a certain output. Five hourly production figures are observed at random from each machine and the results are given below:

Observations	A_1	A_2	A_3
1	25	31	24
2	30	39	30
3	36	38	28
4	38	42	25
5	31	35	28

Use analysis of variance and determine whether the machines are significantly different in their mean speed.

12.5 The following figures related to the number of units of a product sold in five different areas by four salesmen:

Area	Number of units			
	A	B	C	D
1	80	100	95	70
2	82	110	90	75
3	88	105	100	82
4	85	115	105	88
5	75	90	80	65

Is there a significant difference in the efficiency of these salesmen? [*Osmania Univ., MBA, 2006*]

12.6 Four machines A, B, C and D are used to produce a certain kind of cotton fabric. Samples of size 4 with each unit as 100 square meters are selected from the outputs of the machines at random, and the number of flowers in each 100 square meters are counted with the following results:

Machines			
A	B	C	D
8	6	14	20
9	8	12	22
11	10	18	25
12	4	9	23

Do you think that there is significant difference in the performance of the four machines?

[*Kumaon Univ., MBA, 2006*]

Hints and Answers

12.1 Let H_0 : No difference in average sales of three salesmen.

Divide each observation by 100 and use the code data for analysis of variance.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Test Statistic
• Between Samples	10	2	5	$F = \dfrac{5}{2.73}$ $= 1.83$
• Within Samples	30	11	2.73	
Total	40	13		

Since the calculated value of $F = 1.83$ is less than its table value $F = 3.98$ at $df_1 = 2$, $df_2 = 11$, and $\alpha = 0.05$, the null hypothesis is accepted.

12.2 Let H_0 : There is no significant difference in brand preference.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Test Statistic
• Between Samples	168.5	2	84.25	$F = \dfrac{84.25}{22.83}$ $= 3.69$
• Within Samples	205.5	9	22.83	
Total	374.0	11		

Since calculated value of $F = 3.69$ is less than its table value $F = 4.26$ at $df_1 = 2$, $df_2 = 9$, and $\alpha = 0.05$, the null hypothesis is accepted.

12.3 Given total number of observations, $n = 24$; Number of samples, $r = 4$

$$df = n - 1 = 24 - 1 = 23$$

(Between the groups–fertilizers)

$$df_1 = r - 1 = 4 - 1 = 3; \quad df_2 = n - r = 24 - 4 = 20$$

(Within the groups)

$$SSTR = 2940;$$

$$SSE = SST - SSB = 6212 - 2940 = 3272$$

$$MSTR = \frac{SSTR}{df_1} = \frac{2940}{3} = 980;$$

$$MSE = \frac{SSE}{df_2} = \frac{3272}{20} = 163.6$$

$$\therefore \quad F = \frac{MSTR}{MSE} = \frac{980}{163.6} = 5.99$$

Since the calculated value of $F = 5.99$ is more than its table value $F = 3.10$ at $df_1 = 3$, $df_2 = 20$, and $\alpha = 0.05$, the null hypothesis is rejected.

12.4 Let H_0 : Machines are not significantly different in their mean speed.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Test Statistic
• Between Samples	250	2	125	$F = \dfrac{125}{16.66}$ $= 7.50$
• Within Samples	200	12	16.66	
Total	450	14		

Since the calculated value of $F = 7.50$ is more than its table value $F = 3.89$ at $df_1 = 2$, $df_2 = 12$, and $\alpha = 0.05$, the null hypothesis is rejected.

12.5 Let H_0 : No significant difference in the performance of four salesmen.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Test Statistic
• Between Samples	2340	3	780	$F = \dfrac{780}{73.5}$ $= 7.50$
• Within Samples	1176	16	73.5	
Total	3516	19		

Since the calculated value of $F = 10.61$ is greater its table value $F = 3.24$ at $df_1 = 3$, $df_2 = 12$ and $\alpha = 0.05$, the null hypothesis is rejected.

12.6 Let H_0 : Machines do not differ significantly in performance.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Test Statistic
• Between Samples	540.69	3	180.23	$F = \dfrac{180.23}{7.15}$ $= 7.50$
• Within Samples	85.75	12	7.15	
Total	626.44	15		

Since the calculated value of $F = 25.207$ is more than its table value $F = 5.95$ at $df_1 = 3$, $df_2 = 12$, and $\alpha = 0.05$, the null hypothesis is rejected.

12.5 TESTING EQUALITY OF POPULATION MEANS: TWO-WAY CLASSIFICATION

In one-way ANOVA, the partitioning of the total variation in the sample data is done into two components: (i) variation among the samples due to different samples (or treatments) and (ii) variation within the samples due to random error. However, there might be a possibility that some of the variation was due to some other measurable factor rather than due to random error or chance. For example in Example 12.1, part of the variation in price was due to the inability in data collection or condensation of data. If so, this variation was deliberately included in the sum of squares for error (SSE) and therefore caused the mean sum of squares for error (MSE) to be little large. Consequently, F-value would be small and responsible for the rejection of null hypothesis.

Two-way Analysis of Variance: Analysis of variance in which two criteria (or variables) are used to analyse the difference between more than two population means.

The *two-way analysis of variance* can be used to

    (i) explore one criterion (or factor) of interest to partition the sample data so as to remove the unaccountable variation, and arriving at a true conclusion.

    (ii) investigate two criteria (factors) of interest for testing the difference between sample means.

    (iii) consider interaction (if any) between two variables.

In two-way analysis of variance, a *'blocking variable'* is used to remove the undesirable accountable variation. The term *'blocking'* comes from agricultural origin and refers to block of land that might be used to study the growth pattern of varieties of seeds for a given type of land. 'Blocking' is an extension of the idea of pairing observations in hypothesis testing.

The partitioning of total variation in the sample data is shown below:

Blocking: The removal of a source of variation from the error term in the analysis of variance.
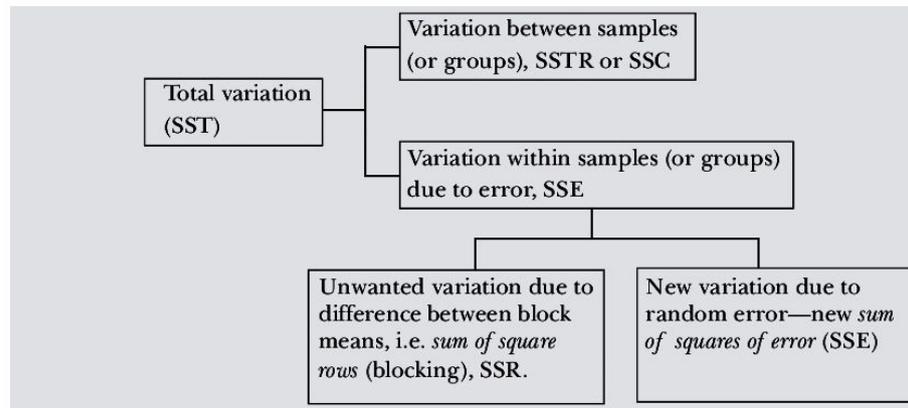
The general ANOVA table for c samples (columns), r blocks and number of observations n is shown in Table 12.7.

Table 12.7 General ANOVA Table for Two-way Classification

Source of Variation	Sum of Square	Degrees of Freedom	Mean Square	Test Statistic
Between columns	SSTR	$c-1$	$MSTR = SSTR/(c-1)$	$F_{treatment} = MSTR/MSE$
Between rows	SSR	$r-1$	$MSR = SSR/(r-1)$	$F_{blocks} = MSR/MSE$
Residual error	SSE	$(c-1)(r-1)$	$MSE = SSE/(c-1)(r-1)$	
Total	SST	$n-1$		

Total variation in the sample data consists of three parts: (i) variation between columns, SSTR; (ii) variation between rows, SSR; and (iii) actual variation due to random error, SSE. That is $\qquad$ SST = SSTR + (SSR + SSE)

The degrees of freedom associated with SST are $cr-1$, where c and r are the number of columns and rows, respectively,

- Degrees of freedom between columns = $c-1$
- Degrees of freedom between rows = $r-1$
- Degrees of freedom for residual error = $(c-1)(r-1) = N-n-c+1$

The test-statistic F for analysis of variance is given by

$$F_{treatment} = MSTR/MSE; \; MSTR > MSE \text{ or } MSE/MSTR; \; MSE > MSTR$$
$$F_{blocks} \; MSR/MSE; \; MSR > MSE \text{ or } MSE/MSR; \; MSE > MSR$$

Decision Rules

- If $F_{cal} < F_{table}$, then accept null hypothesis, H_0.
- Otherwise reject H_0.

Example 12.4: The following table gives the number of refrigerators sold by 4 salesmen in three months May, June and July:

Month	Salesman			
	A	*B*	*C*	*D*
May	50	40	48	39
June	46	48	50	45
July	39	44	40	39

Is there a significant difference in the sales made by the four salesmen? Is there a significant difference in the sales made during different months?   [*Delhi Univ., M.Com., 2006*]

Solution: Let us take the null hypothesis that there is no significant difference between sales made by the four salesmen during different months.

The given data are coded by subtracting 40 from each observation. Calculations for a two-criteria—month and salesman—analysis of variance are shown in Table 12.8.

Randomized Block Design: A two-way analysis of variance designed to eliminate any assignable variation from the analysis.

Table 12.8   Two-way ANOVA Table

Month	Salesman								Row
	$A(x_1)$	x_1^2	$B(x_2)$	x_2^2	$C(x_3)$	x_3^2	$D(x_4)$	x_4^2	Sum
May	10	100	0	0	8	64	−1	1	17
June	6	36	8	64	10	100	5	25	29
July	−1	01	4	16	0	0	−1	1	2
Column sum	15	137	12	80	18	164	3	27	48

T = Sum of all observations in three samples of months = 48

$$CF = \text{Correction factor} = \frac{T^2}{n} = \frac{(48)^2}{12} = 192$$

$$SSTR = \text{Sum of squares between salesmen (columns)}$$

$$= \left\{ \frac{(15)^2}{3} + \frac{(12)^2}{3} + \frac{(18)^2}{3} + \frac{(3)^2}{3} \right\} - 192$$

$$= (75 + 48 + 108 + 3) - 192 = 42$$

$$SSR = \text{Sum of squares between months (rows)}$$

$$= \left\{ \frac{(17)^2}{4} + \frac{(29)^2}{4} + \frac{(2)^2}{4} \right\} - 192$$

$$= (72.25 + 210.25 + 1) - 192 = 91.5$$

$$SST = \text{Total sum of squares}$$

$$= (\Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2) - CF$$

$$= (137 + 80 + 164 + 27) - 192 = 216$$

$$SSE = SST - (SSC + SSR) = 216 - (42 + 91.5) = 82.5$$

The total degrees of freedom are $df = n - 1 = 12 - 1 = 11$. Thus,

$$df_c = c - 1 = 4 - 1 = 3, \quad df_r = r - 1 = 3 - 1 = 2; \quad df = (c - 1)(r - 1) = 3 \times 2 = 6$$

and

$$MSTR = SSTR/(c - 1) = 42/3 = 14,$$

$$MSR = SSR/(r - 1) = 91.5/2 = 45.75$$

$$MSE = SSE/(c - 1)(r - 1) = 82.5/6 = 13.75$$

The ANOVA table is shown in Table 12.9.

Table 12.9   Two-way ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	Variance Ratio
• Between salesmen	42.0	3	14.00	$F_{treatment} = 14/13.75$ = 1.018
• Between months	91.5	2	45.75	$F_{block} = 45.75/13.75$ = 3.327
• Residual error	82.5	6	13.75	
Total	216	11		

(a) Since the calculated value of $F_{treatment} = 1.018$ is less than its table value, $F = 4.75$ at $df_1 = 3$, $df_2 = 6$, and $\alpha = 0.05$, the null hypothesis is accepted. Hence, it may be concluded that sales made by the salesmen do not differ significantly.

(b) Since the calculated value of $F_{block} = 3.327$ is less than its table value, $F = 5.14$ at $df_1 = 2$, $df_2 = 6$, and $\alpha = 0.05$, the null hypothesis is accepted. Hence, it may be concluded that sales made during different months do not differ significantly.

Example 12.5: To study the performance of three detergents and three different water temperatures, the following 'whiteness' readings were obtained with specially designed equipment:

Water Temperature	Detergent A	Detergent B	Detergent C
Cold water	57	55	67
Warm water	49	52	68
Hot water	54	46	58

Perform a two-way analysis of variance, using 5 per cent level of significance.

[Osmania Univ., MBA, 2006]

Solution: Let us take the null hypothesis that there is no significant difference in the performance of three detergents due to water temperature and vice-versa.

The data are coded by subtracting 50 from each observation. The data in coded form are shown in Table 12.10.

Table 12.10   Coded Data

Water Temperature	Detergents $A(x_1)$	x_1^2	$B(x_2)$	x_2^2	$C(x_3)$	x_3^2	Row Sum
Cold water	7	49	5	25	17	289	29
Warm water	-1	01	2	04	18	324	19
Hot water	4	16	-4	16	8	64	8
Column sum	10	66	3	45	43	677	56

$$T = \text{Sum of all observations in three samples of detergents} = 56$$

$$CF = \text{Correction factor} = \frac{T^2}{n} = \frac{(56)^2}{9} = 348.44$$

$$SSTR = \text{Sum of squares between detergents (columns)}$$

$$= \left\{ \frac{(10)^2}{3} + \frac{(3)^2}{3} + \frac{(43)^2}{3} \right\} - CF$$

$$= 33.33 + 3 + 616.33 - 348.44 = 304.22$$

$$SSR = \text{Sum of squares between water temperature (rows)}$$

$$= \left\{ \frac{(29)^2}{3} + \frac{(19)^2}{3} + \frac{(8)^2}{3} \right\} - CF$$

$$= (280.33 + 120.33 + 21.33) - 348.44 = 73.55$$

$$SST = \text{Total sum of squares}$$

$$= (\Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2) - CF$$

$$= (66 + 45 + 677) - 348.44 = 439.56$$

$$SSE = SST - (SSTR + SSR) = 439.56 - (304.22 + 73.55) = 61.79$$

Thus
$$MSTR = SSTR/(c - 1) = 304.22/2 = 152.11;$$

$$MSR = SSR/(r - 1) = 73.55/2 = 36.775$$

$$MSE = SSE/(c - 1)(r - 1) = 61.79/4 = 15.447$$

Table 12.11 Two-way ANOVA Table

Source of Variation	Sum of Squares	Degree of Freedom	Mean Squares	Variance Ratio
Between detergents (columns)	304.22	2	152.110	$F_{treatment} = 152.11/15.447$ $= 9.847$
Between temp. (rows)	73.55	2	36.775	$F_{block} = 36.775/15.447$ $= 2.380$
Residual error	61.79	4	15.447	
Total	439.56	8		

(a) Since calculated value of $F_{treatment} = 9.847$ is greater than its table value, $F = 6.94$, at $df_1 = 2$, $df_2 = 4$, and, $\alpha = 0.05$, the null hypothesis is rejected. Hence, it may be concluded that there is significant difference between the performance of the three detergents.

(b) Since the calculated value of $F_{block} = 2.380$ is less than its table value, $F = 6.94$, at $df_1 = 2$, $df_2 = 4$, and $\alpha = 0.05$, the null hypothesis is accepted. Hence, it may be concluded that the water temperature do not make a significant difference in the performance of the detergent.

Conceptual Questions 12A

1. What are some of the criteria used in the selection of a particular hypothesis testing procedure?

2. What are the major assumptions of ANOVA?

3. Under what conditions should the one-way ANOVA F-test to be selected to examine the possible difference in the means of independent populations?

4. How is analysis of variance technique helpful in solving business problems? Illustrate your answer with suitable examples. *[Kumaon Univ., MBA, 2004]*

5. Distinguish between one-way and two-way classifications to test the equality of population means.

6. What is meant by the term analysis of variance? What types of problems are solved using ANOVA? Explain.

7. Describe the procedure for performing the test of hypothesis in the analysis of variance. What is the basic assumption underlying this test?

8. What is meant by the critical value used in the analysis of variance? How is it found?

9. How is the F-distribution related to the student's *t*-distribution and the chi-square distribution? What important hypothesis can be tested by the F-distribution?

10. Discuss the components of total variation when samples are selected in blocks.

11. Define the terms treatment, error— 'within' 'between' and the context in which these are used.

12. Explain the sum-of-square principle.

13. Explain how the total deviation is partitioned into the treatment deviation and the error deviation.

14. Does the quantity MSTR/MSE follow an F-distribution when the null hypothesis of ANOVA is false? Explain.

Self-practice Problems 12B

12.7 A tea company appoints four salesmen A, B, C and D, and observes their sales in three seasons—summer, winter and monsoon. The figures (in lakhs) are given in the following table:

Season	Salesman				Total
	A	B	C	D	
Summer	36	36	21	35	128
Winter	28	29	31	32	120
Monsoon	26	28	29	29	112
Totals	90	93	81	96	360

(a) Do the salesmen significantly differ in performance?
(b) Is there significant difference between the seasons?

[Kolkatta Univ., M.Com., 2004; Kolkatta Univ., M.Com., 2006]

12.8 Perform a two-way ANOVA on the data given below:

Plots of Land	Treatment			
	A	B	C	D
1	38	40	41	39
2	45	42	49	36
3	40	38	42	42

Use the coding method for subtracting 40 from the given numbers.

12.9 The following data represent the production per day turned out by 5 different workers using 4 different types of machines:

Workers	Machine Type			
	A	B	C	D
1	44	38	47	36
2	46	40	52	43
3	34	36	44	32
4	43	38	46	33
5	38	42	49	39

(a) Test whether the mean productivity is the same for the different machine types.
(b) Test whether the 5 men differ with respect to mean productivity. *[Madras Univ., M.Com., 2005]*

12.10 The following table gives the number of units of production per day turned out by four different types of machines:

Employees	Type of Machine			
	M_1	M_2	M_3	M_4
E_1	40	36	45	30
E_2	38	42	50	41
E_3	36	30	48	35
E_4	46	47	52	44

Using analysis of variance (a) test the hypothesis that the mean production is same for four machines and (b) test the hypothesis that the employees do not differ with respect to mean productivity.

[Osmania Univ., M.Com., 2007]

12.11 In a certain factory, production can be accomplished by four different workers on five different types of machines. A sample study, in the context of a two-way design without repeated values, is being made with two fold objectives of examining whether the four workers differ with respect to mean productivity and whether the mean productivity is the same for the five different machines. The researcher involved in this study reports while analysing the gathered data as under:

(a) Sum of squares for variance between machines = 35.2

(b) Sum of squares for variance between workmen = 53.8

(c) Sum of squares for total variance = 174.2

Set up ANOVA table for the given information and draw the inference about variance at 5 per cent level of significance.

12.12 Apply the technique of analysis of variance of the following data showing the yields of 3 varieties of a crop each from 4 blocks, and test whether the average yields of the varieties are equal or not. Also test equality of the block means.

Varieties	Blocks			
	I	II	III	IV
A	4	8	6	8
B	5	5	7	8
C	6	7	9	5

12.13 Three varieties of potato are planted each on four plots of land of the same size and type, each variety is treated with four different fertilizers. The yield in tonnes is as follows:

Fertilizer	Variety		
	V_1	V_2	V_3
F_1	164	172	174
F_2	155	157	147
F_3	159	166	158
F_4	158	157	153

Perform an analysis of variance and show whether (a) there is any significant difference between the average yield of potatoes due to different fertilizers being used, and (b) there is any difference in the average yield of potatoes of different varieties.

Hints and Answers

12.7 Let H_0 : No significant difference between sales by salesmen and that of seasons.

Decoding the data by subtracting 30 from each figure.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Variance Ratio
Between salesmen (column)	42	3	14	$F_1 = \dfrac{22.67}{14}$ $= 1.619$
Between seasons (row)	32	2	16	$F_2 = \dfrac{22.67}{16}$ $= 1.417$
Residual error	136	6	22.67	
Total	210	11		

• Since $F_1 = 1.619 < F_{0.05\ (6,\ 3)} = 4.76$, accept null hypothesis.

• Since $F_2 = 1.417 < F_{0.05(6,\ 2)} = 5.14$, accept null hypothesis.

12.8

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Variance Ratio
Between columns	42	3	14	$F_1 = \dfrac{14}{10.67}$ $= 1.312$
Between rows	26	2	13	$F_2 = \dfrac{13}{10.67}$ $= 1.218$
Residual error	64	6	10.67	
Total	132	11		

(a) $F_1 = 1.312 < F_{0.05\ (3,\ 6)} = 4.76$, accept null hypothesis.

(b) $F_2 = 1.218 < F_{0.05\ (2,\ 6)} = 5.14$, accept null hypothesis.

12.9 Let H_0 : (a) Mean productivity is same for all machines

(b) Men do not differ with respect to mean productivity

Decoding the data by subtracting 40 from each figure.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Variance Ratio
Between machine type	338.8	3	112.933	$F_1 = \dfrac{112.933}{6.142}$ $= 18.387$
Between workers	161.5	4	40.375	$F_2 = \dfrac{40.375}{6.142}$ $= 6.574$
Residual error	73.7	12	6.142	
Total	574	19		

(a) $F_{0.05} = 3.49$ at $df_1 = 3$ and $df_2 = 12$. Since the calculated value $F_1 = 18.387$ is greater than the table value, the null hypothesis is rejected.

(b) $F_{0.05} = 3.26$ at $df_1 = 4$ and $df_2 = 12$. Since the calculated value $F_2 = 6.574$ is greater than the table value, the null hypothesis is rejected.

12.10 Let H_0 : (a) Mean production does not differ for all machines
(b) Employees do not differ with respect to mean productivity

Decoding the data by subtracting 40 from each figure.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Variance Ratio
Between machine	312.5	3	104.17	$F_1 = \dfrac{104.17}{10.72}$ $= 9.72$
Between employees	1266.0	3	88.67	$F_2 = \dfrac{88.67}{10.72}$ $= 8.27$
Residual error	96.5	9	10.72	
Total	675.0	15		

(a) $F_{0.05} = 3.86$ at $df_1 = 3$ and $df_2 = 9$. Since the calculated value $F_1 = 9.72$ is more than its table value, reject the null hypothesis.

(b) $F_{0.05} = 3.86$ at $df_1 = 3$ and $df_2 = 9$. Since the calculated value $F_2 = 8.27$ is more than its table value, reject the null hypothesis

12.11 Let H_0 : (a) Workers do not differ with respect to their mean productivity.
(b) Mean productivity of all machines is the same.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Variance Ratio
Between machine	35.2	4	8.8	$F_1 = \dfrac{8.8}{7.1}$ $= 1.24$
Between workmen	53.8	3	17.93	$F_2 = \dfrac{17.93}{7.1}$ $= 2.53$
Residual error	85.2	12	7.1	
Total	174.2	19		

(a) The calculated value of $F_1 = 1.24$ is less than its table value $F_{0.05} = 3.25$ at $df_1 = 4$ and $df_2 = 12$, hence the null hypothesis is accepted.

(b) The calculated value of $F_2 = 2.53$ is less than its table value $F_{0.05} = 3.49$ at $df_1 = 3$ and $df_2 = 12$, hence the null hypothesis accepted.

12.12 Let H_0: (a) Mean yields of the varieties are equal
(b) Block means are equal

Decoding the data by subtracting 5 from each figure.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Variance Ratio
Between blocks	9.67	3	3.22	$F_1 = \dfrac{3.22}{2.80}$ $= 1.15$
Between varieties	0.5	2	0.25	$F_2 = \dfrac{2.80}{0.25}$ $= 11.20$
Residual error	16.83	6	2.80	
Total	27.00	12		

(a) Since the calculated value $F_1 = 1.15$ is less than its table value $F_{0.05} = 4.757$ at $df = (3, 6)$, the null hypothesis is accepted.

(b) Since the calculated value $F_2 = 11.20$ is less than its table value $F_{0.05} = 19.33$ at $df = (6, 2)$, the null hypothesis is accepted.

12.13 Let H_0 : (a) No significant difference in the average yield of potatoes due to different fertilizers.
(b) No significant difference in the average yield of the three varieties of potatoes.

Decoding the data by subtracting 158 from each figure.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Variance Ratio
Between varieties	56	2	28	$F_1 = \dfrac{28}{18}$ $= 1.55$
Between fertilizers	498	3	166	$F_2 = \dfrac{166}{18}$ $= 9.22$
Residual error	108	6	18	
Total	$\overline{662}$	$\overline{11}$		

(a) $F_{cal} = 1.55$ is less than its table value $F_{0.05} = 5.14$ at $df = (2, 6)$, the null hypothesis is accepted.

(b) $F_{cal} = 9.22$ is more than its table value $F_{0.05} = 4.67$ at $df = (3, 6)$, the null hypothesis is rejected.

Formulae Used

1. **One-way analysis of variance**
   - Grand sample mean
   $$\overline{x} = \frac{\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{r} x_{ij}}{n}, n = n_1 + n_2 + \dots + n_r$$
   - Correction factor $CF = \dfrac{T^2}{n}$
   - Total sum of squares
   $$SST = \sum_{i=1}^{k}\sum_{j=1}^{r}(x_{ij} - \overline{\overline{x}})^2 = \sum_i\sum_j x_{ij}^2 - CF$$
   - Sum of squares of variations between samples due to treatment
   $$SSTR = \sum_{j=1}^{r} n_j(\overline{x}_j - \overline{\overline{x}})^2 = \frac{1}{n_j}\sum_{j=1}^{r} x_j^2 - CF$$
   - Sum of squares of variations within samples or error sum of squares
   $$SSE = \sum_i^k\sum_j^r(x_{ij} - \overline{x}_i)^2 = SST - SSTR$$
   - Mean square between samples due to treatments
   $$MSTR = \frac{SSTR}{r-1}$$
   - Mean square within samples due to error
   $$MSE = \frac{SSE}{n-r}$$
   - Test statistic for equality of k population means
   $$F = \frac{MSTR}{MSE}$$
   - Degrees of freedom
   Total df $(n-1)$ = Treatment df $(r-1)$ + Random error df $(n-r)$

2. **Two-way analysis of variance**
   - Total sum of squares
   $$SST = \sum_{j=1}^{r}\sum_{i=1}^{k} x_{ij}^2 - n(\overline{\overline{x}})^2$$
   - Sum of squares of variances between columns due to treatments
   $$SSTR = \sum_{j=1}^{r}(\overline{x}_j)^2 - n(\overline{\overline{x}})^2$$
   - Sum of squares between rows due to blocks
   $$SSR = \sum_{i=1}^{k}(\overline{x}_i)^2 - n(\overline{\overline{x}})^2$$
   - Sum of squares due to error
   $$SSE = SST - (SSTR + SSR)$$
   - Degrees of freedom
   $$df_c = c - 1; df_r = (r - 1)$$
   df(residual error) = Blocks df + Treatments df
   $$= (r-1)(c-1)$$
   - Mean squares between columns due to treatment
   $$MSTR = \frac{SSTR}{c-1}$$
   - Mean square between rows due to blocks
   $$MSR = \frac{SSR}{r-1}$$
   - Mean square of residual error
   $$MSE = \frac{SSE}{(c-1)(r-1)}$$
   - Test statistic
   $$F_1 = \frac{MSTR}{MSE}; F_2 = \frac{MSR}{MSE}$$
   provided numerator is bigger than denominator.