Chapter 13
The Black-Scholes-Merton Model

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13.1. The Black-Scholes option pricing model assumes that the probability distribution of the stock price in one year(or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normal distributed.

13.2. The standard deviation of the percentage price change in time δt is $\sigma\sqrt{\delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

13.3. The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preference. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This samplifies the analysis. In a risk-neutral world, all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

13.4. In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.3^2/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$p = 50e^{-0.1\times 0.25}N(-0.0917) - 50N(-0.2417)$$
$$= 50 \times 0.4634e^{-0.1\times 0.25} - 50 \times 0.4045 = 2.37$$

or \$2.37.

13.5. In this case we must subtract the present value of the dividend from the stock price before using Black-Scholes. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667\times 0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$ and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times 0.25} - 48.52N(-0.0414)$$
$$= 50 \times 0.5432e^{-0.1\times 0.25} - 48.52 \times 0.4835 = 3.03$$

or \$3.03.

13.6. The implied volatility is the volatility that makes the Black-Scholes price of an option equal to its market price. It is calculated using an iterative procedure.

13.7. In this case, $\mu = 0.15$ and $\sigma = 0.25$. From equation (13.7), the probability distribution for the rate of return over a 2-year period with continuous compounding is

$$\phi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25}{\sqrt{2}}\right)$$

i.e.,

$$\phi(0.11875, 0.1768)$$

The expected value of the ruturn is 11.875% per annum and the standard deviation is 17.68% per annum.

13.8. (a) The required probability is the probability of the stock price being above \$40 in six months' time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \phi\left[\ln 38 + \left(0.16 - \frac{0.35^2}{2}\right)0.5, 0.35\sqrt{0.5}\right]$$

i.e.,

$$\ln S_T \sim \phi(3.687, 0.247)$$

Since $\ln 40 = 3.689$, the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

From normal distribution tables $N(0.008) = 0.5032$, so that the required probability is 0.4968. In general the required probability is $N(d_2)$.

(b) In this case, the required probability is the probability of the stock price being less than \$40 in six months' time. It is

$$1 - 0.4968 = 0.5032$$

13.9. From equation (13.3),

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

Thus the 95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T's are therefore

$$e^{\ln S_0 + (\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}}$$

and

$$e^{\ln S_0 + (\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

i.e.,

$$S_0 e^{(\mu - \sigma^2/2t)T - 1.96\sigma\sqrt{T}}$$

and

$$S_0 e^{(\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

13.10. The statement is misleading in that a certain sum of money, say $1000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.

The average of the returns realized in each year is always greater than the return per annum(with annual compounding) realized over 10 years.The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

13.11. (a) At time t, the expected value of $\ln S_T$ is, from equation (13.3)

$$\ln S + \left(\mu - \frac{\sigma^2}{2}\right)(T - t)$$

In a risk-neutral world, the expected value of $\ln S_T$ is therefore:

$$\ln S + \left(r - \frac{\sigma^2}{2}\right)(T - t)$$

Using risk-neutral valuation, the value of the security at time t is:

$$e^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T - t)\right]$$

(b) If:

$$f = e^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]$$

then

$$\begin{aligned}
\frac{\partial f}{\partial t} &= re^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] - e^{-r(T-t)}\left(r - \frac{\sigma^2}{2}\right)\\
\frac{\partial f}{\partial S} &= \frac{e^{-r(T-t)}}{S}\\
\frac{\partial^2 f}{\partial S^2} &= -\frac{e^{-r(T-t)}}{S^2}
\end{aligned}$$

The left-hand side of the Black-Scholes-Merton differential equation is

$$\begin{aligned}
&e^{-r(T-t)}\left[r\ln S + r\left(r - \frac{\sigma^2}{2}\right)(T-t) - \left(r - \frac{\sigma^2}{2}\right) + r - \frac{\sigma^2}{2}\right]\\
=&re^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]\\
=&rf
\end{aligned}$$

Hence equation (13.16) is satisfied.

13.12. This problem is related to Problem 12.10.

(a) If $G(S,t) = h(t,T)S^n$, then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$, where $h_t = \partial h/\partial t$. Substituting into the B-S-M differential equation, we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

(b) The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$.

(c) The equation

$$h(t,T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

satisfies the boundary condition, since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can also solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)]t + k$$

where k is a constant. Since $\ln h = 0$ when $t = T$, it follows that

$$k = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)]T$$

So that

$$\ln h = [-r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T-t)$$

or

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

13.13. In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.3$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.3\sqrt{0.25}} = 0.5365$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.3865$$

The price of the European call is

$$52N(0.5365) - 50e^{-0.12\times0.25}N(0.3865)$$
$$= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504 = 5.06$$

or \$5.06.

13.14. In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$
$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$70e^{-0.05\times0.5}N(0.0809) - 69N(-0.1666)$$
$$= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338 = 6.40$$

or \$6.40.

13.15. Using the notation in Section 13.12, $D_1 = D_2 = 1$, and

$$K[1 - e^{-r(t_2-t_1)}] = 65[1 - e^{-0.1\times0.25}] = 1.60$$
$$K[1 - e^{-r(T-t_2)}] = 65[1 - e^{-0.1\times0.1667}] = 1.07$$

Since

$$D_1 < K[1 - e^{-r(t_2-t_1)}]$$

and

$$D_2 < K[1 - e^{-r(T-t_2)}]$$

So it is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

13.16. In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be caculated using an interative procedure.

A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.397 or 39.7% per annum.

13.17. (a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(b)

$$\begin{aligned} N'(d_1) &= N'(d_2 + \sigma\sqrt{T-t}) \\ &= \frac{1}{\sqrt{2\pi}} exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] \\ &= N'(d_2) exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] \end{aligned}$$

Because

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

It follows that

$$exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Which is the required result.

(c)

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

(d)

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$d_1 - d_2 = \sigma\sqrt{T-t}$$

$$\begin{aligned}\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} &= \frac{\partial}{\partial t}(\sigma\sqrt{T-t}) \\ &= -\frac{\sigma}{2\sqrt{T-t}}\end{aligned}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black-Scholes formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N(d_2)\frac{\partial d_2}{\partial S}$$

From the results in (b) and (c), it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\begin{aligned}\frac{\partial^2 c}{\partial S^2} &= N'(d_1)\frac{\partial d_1}{\partial S} \\ &= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}\end{aligned}$$

From the result in (d) and (e)

$$\begin{aligned}&\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} \\ =& -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} + rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \\ =& r[SN(d_1) - Ke^{-r(T-t)}N(d_2)] \\ =& rc\end{aligned}$$

This shows that the Black-Scholes formula for a call option does indeed satisfy the Black-Scholes differencial equation.

(g) If $S \geq K$, then $d_1, d_2 \to +\infty$ as $t \to T$, $N(d_1) = N(d_2) = 1$. From (d), $c = S - K$. If $S < K$, then $d_1, d_2 \to -\infty$ as $t \to T$, $N(d_1) = N(d_2) = 0$. From (d), $c = 0$. Therefore c satisfies the boundary condition for a European call option, i.e., that $c = \max(S - K, 0)$ as $t \to T$.

13.18. From the Black-Scholes equations

$$p + S_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0$$

Because $1 - N(-d_1) = N(d_1)$, this is

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

Also:

$$c + Ke^{-rT} = S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$

Because $1 - N(d_2) = N(-d_2)$, this is also

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

The Black-Scholes equations are therefore consistent with put-call parity.

13.19. This problem naturally leads on to the material in Chapter 18 on volatility smiles. Using DerivaGem we obtain the following table of implied volatilities:

	Maturity (months)		
Strike price ($)	3	6	12
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

The option prices are not exactly consistent with Black-Scholes. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock.

13.20. Black's approach in effct assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time T.

It appears that Black's approach understates the ture option value. This is because the holder of the option has more alternative strategies for decideing when to exercise the option than the two alternatives implicitly assumed by the approach. These alternatives add value to the option.

13.21. With the notation in the text, $D_1 = D_2 = 1.50$, $t_1 = 1/3$, $t_2 = 5/6$, $T = 1.25$, $r = 0.08$ and $K = 55$

$$K[1 - e^{-r(T-t_2)}] = 55[1 - e^{-0.08\times 0.4167}] = 1.80$$

Hence

$$D_2 < K[1 - e^{-r(T-t_2)}]$$

Also:

$$K[1 - e^{-r(t_2-t_1)}] = 55[1 - e^{-0.08\times 0.5}] = 2.16$$

Hence:

$$D_1 < K[1 - e^{-r(t_2-t_1)}]$$

It follows from the conditions established in Section 13.12 that the option should never be exercised early.

The present value of the dividends is

$$1.5e^{-0.08\times 1/3} + 1.5e^{-0.08\times 5/6} = 2.8638$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.8638 = 47.1362, K = 55, \sigma = 0.25, r = 0.08, T = 1.25$$

$$d_1 = \frac{\ln(47.1362/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, N(d_2) = 0.3692$$

and the call price is

$$47.1362 \times 0.4783 - 55e^{-0.08\times 1.25} \times 0.3692 = 4.17$$

or \$4.17.

13.22. The probability that the call option will be exercised is the probability that $S_T \geq K$ where S_T is the stock price at time T. In a risk neutral world,

$$\ln S_T \sim \phi\left[\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

The probability that $S_T \geq K$ is the same as the probability that $\ln S_T \geq \ln K$. This is

$$1 - N\left(\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) = N\left(\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) = N(d_2)$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T \geq K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security at time t is

$$100e^{-rT}N(d_2)$$

13.23. If $f = S^{-2r/\sigma^2}$, then

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial S} = -\frac{2r}{\sigma^2} S^{-\frac{2r}{\sigma^2}-1}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{2r}{\sigma^2}\left(\frac{2r}{\sigma^2}+1\right) S^{-\frac{2r}{\sigma^2}-2}$$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rS^{-2r/\sigma^2} = rf$$

This show that the Black-Scholes equation is satisfied. S^{-2r/σ^2} could therefore be the price of a traded security.

13.24. No. This is mainly because when markets are efficient the impact of dilution from executive stock options or warrants is reflected in the stock price as soon as they are announced and does not need to be taken into account again when the options are valued.

13.25. In this case, $S_0 = K = 50$, $r = 0.05$, $\sigma = 0.25$, $T = 5$, $N = 10,000,000$ and $M = 3,000,000$. From Section 13.10, the value of each employee option is the value of $N/(N+M)$ regular call options on the company's stock, and the total cost is M times this.

Thus applying Black-Scholes Formula,

$$d_1 = \frac{\ln(50/50) + (0.05 + 0.25^2/2)5}{0.25\sqrt{5}} = 0.7267,$$
$$d_2 = d_1 - 0.25\sqrt{5} = 0.1677.$$

The regular European call price is

$$c = 50N(0.7267) - 50e^{-0.05\times 5}N(0.1677),$$
$$= 16.2517.$$

Therefore the cost to the company of the employee stock option issue is

$$M \cdot \frac{N}{N+M} \cdot c$$
$$=3,000,000 \times \frac{10,000,000}{10,000,000 + 3,000,000} \times 16.2517,$$
$$=37,503,923.08,$$

or \$37,503,923.08.

13.26. In this case, $S_0 = 50$, $\mu = 0.18$, $\sigma = 0.3$. The probability distribution of the stock price in two years, S_T, is lognormal and is, from equation (13.3), given by:

$$\ln S_T \sim \phi\left[\ln 50 + \left(0.18 - \frac{0.09}{2}\right)2, 0.3\sqrt{2}\right]$$

i.e.,

$$\ln S_T \sim \phi(4.18, 0.42)$$

The mean stock price is from equation (13.4)

$$50e^{0.18\times2} = 50e^{0.36} = 71.67$$

and the standard deviation is, from equation (13.5)

$$50^2e^{2\times0.18}\sqrt{e^{0.09\times2}-1} = 31.83$$

95% confidence intervals for $\ln S_T$ are

$$4.18 - 1.96\times0.42 \quad \text{and} \quad 4.18 + 1.960.42$$

i.e.,

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

i.e.,

$$28.52 \quad \text{and} \quad 150.44$$

13.27. The calculation are shown in the table below

$$\Sigma u_i = 0.09471 \quad \Sigma u_i^2 = 0.01145$$

and an estimate of standard deviation of weekly return is :

$$\sqrt{\frac{0.01145}{13} - \frac{0.09471^2}{14\times13}} = 0.02884$$

The volatility per annum of $0.0288\sqrt{52} = 0.2079$, or 20.79%. The standard error of this estimate is

$$\frac{0.2079}{\sqrt{2\times14}} = 0.0393$$

or 3.93% per annum.

Computation of Volatility			
Week	*Closing stock price* ($)	*Price relative* S_i/S_{i-1}	*Weekly return* $u_i = \ln(S_i/S_{i-1})$
1	30.2		
2	32.0	1.0596	0.0579
3	31.1	0.9719	−0.0285
4	30.1	0.9678	−0.0327
5	30.2	1.0033	0.0033
6	30.3	1.0033	0.0033
7	30.6	1.0099	0.0099
8	33.0	1.0784	0.0755
9	32.9	0.9970	−0.0030
10	33.0	1.0030	0.0030
11	33.5	1.0152	0.0151
12	33.5	1.0000	0.0000
13	33.7	1.0060	0.0060
14	33.5	0.9941	−0.0059
15	33.2	0.9910	−0.0090

13.28. (a) The expected value of the security is $E[S_T^2]$. From equations (13.4) and (13.5), at time t:

$$E(S_T) = Se^{\mu(T-t)}$$

$$\text{var}(S_T) = S^2 e^{2\mu(T-t)}(e^{\sigma^2(T-t)} - 1)$$

Since $\text{var}(S_T) = E(S_T^2) - [E(S_T)]^2$, it follows that $E(S_T^2) = \text{var}(S_T) + [E(S_T)]^2$, so that

$$\begin{aligned} E(S_T^2) &= S^2 e^{2\mu(T-t)} + S^2 e^{2\mu(T-t)}(e^{\sigma^2(T-t)} - 1) \\ &= S^2 e^{(2\mu+\sigma^2)(T-t)} \end{aligned}$$

In a risk-neutral world $\mu = r$, so that

$$\hat{E}(S_T^2) = S^2 e^{(2r+\sigma^2)(T-t)}$$

Using risk-neutral valuation,the value of the derivative security at time t is

$$\begin{aligned} &e^{-r(T-t)}\hat{E}(S_T^2) \\ &= S^2 e^{(2r+\sigma^2)(T-t)} e^{-r(T-t)} \\ &= S^2 e^{(r+\sigma^2)(T-t)} \end{aligned}$$

(b) If

$$f = S^2 e^{(r+\sigma^2)(T-t)}$$

then

$$\begin{aligned}\frac{\partial f}{\partial t} &= -S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)}\\ \frac{\partial f}{\partial S} &= 2Se^{(r+\sigma^2)(T-t)}\\ \frac{\partial^2 f}{\partial S^2} &= 2e^{(r+\sigma^2)(T-t)}\end{aligned}$$

The left-hand side of the Black-Scholses differential equation is:

$$\begin{aligned}&-S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)} + 2rS^2e^{(r+\sigma^2)(T-t)} + \sigma^2S^2e^{(r+\sigma^2)(T-t)}\\ =&rS^2e^{(r+\sigma^2)(T-t)}\\ =&rf\end{aligned}$$

Hence the Black-Scholses equation is satisfied.

13.29. In this case $S_0 = 30$, $K = 29$, $r = 0.05$, $\sigma = 0.25$ and $T = 1/3$.

$$\begin{aligned}d_1 &= \frac{\ln(30/29) + (0.05 + 0.25^2/2)/3}{0.25\sqrt{1/3}} = 0.4225\\ d_2 &= d_1 - 0.25\sqrt{1/3} = 0.2782\end{aligned}$$

$$\begin{aligned}N(0.4225) = 0.6637 \quad N(0.2782) = 0.6096\\ N(-0.4225) = 0.3363 \quad N(-0.2782) = 0.3904\end{aligned}$$

(a) The European call price is

$$30 \times 0.6637 - 29e^{-0.05/3} \times 0.6096 = 2.52$$

or \$2.52.

(b) The American call price is the same as the European call price. It is \$2.52.

(c) The European put price is

$$29e^{-0.05/3} \times 0.3904 - 30 \times 0.3363 = 1.05$$

or \$1.05.

(d) Put-call parity states that:

$$c + Ke^{-rT} = p + S_0$$

In this case $c = 2.52$, $S_0 = 30$, $K = 29$, $p = 1.05$ and $e^{-rT} = 0.9835$ and it is easy to verify that the relationship is satisfied.

13.30. (a) The present value of the dividend must be subtracted from the stock price. This gives a new stock price of:

$$30 - 0.5e^{-0.125\times0.05} = 29.5031$$

and

$$d_1 = \frac{\ln(29.5031/29) + (0.05 + 0.25^2/2)/3}{0.25\sqrt{1/3}} = 0.3068$$
$$d_2 = d_1 - 0.25\sqrt{1/3} = 0.1625$$

$$N(d_1) = 0.6205, \quad N(d_2) = 0.5645$$

The price of the option is thherefore

$$29.5031 \times 0.6205 - 29e^{-0.05/3} \times 0.5645 = 2.21$$

or \$2.21.

(b) Since

$$N(-d_1) = 0.3795; \quad N(-d_2) = 0.4355$$

the value of the option when it is a European put is

$$29e^{-0.05/3} \times 0.4355 - 29.5031 \times 0.3795 = 1.22$$

or \$1.22.

(c) If t_1 denotes the time when the dividend is paid:

$$K[1 - e^{-r(T-t_1)}] = 29(1 - e^{-0.05\times0.2083}) = 0.3005$$

This is less than the dividend. Hence the option should be exercised immediately before the ex-dividend date for a sufficiently high value of the stock price.

13.31. We first value the option assuming that it is not exercised early, we set the time to maturity equal to 0.5. There is a dividend of 0.4 in 2 months and 5 months. Other parameters are $S_0 = 18$, $K = 20$, $r = 10\%$, $\sigma = 30\%$. DerivaGem gives the price as 0.7947. We next value the option assuming that it is exercised at the five-month point just before the final dividend. DerivaGem gives the price as 0.7668. The price given by Black's approximation is therefore 0.7947. DerivaGem also shows that the correct American option price calculate with 100 time steps is 0.8243.

It is never optimal to exercise the option immediately before the first ex-dividend date when

$$D_1 \leq K[1 - e^{-r(t_2-t_1)}]$$

Where D_1 is the size of the frist dividend, and t_1 and t_2 are the times of the first and second dividend respectively. Hence we must have:

$$D_1 \leq 20[1 - e^{-0.1\times0.25}]$$

that is,

$$D_1 \leq 0.494$$

It is never optimal to exercise the option immediately before the second ex-dividend date when:

$$D_2 \leq K[1 - e^{-r(T-t_2)}]$$

Where D_2 is the size of the second dividend. Hence we must have:

$$D_2 \leq 20[1 - e^{-0.1\times 0.0833}]$$

that is,

$$D_2 \leq 0.166$$

It follows that the dividend can be as high as 16.6 cents per share without the American option being worth more than the corresponding European option.