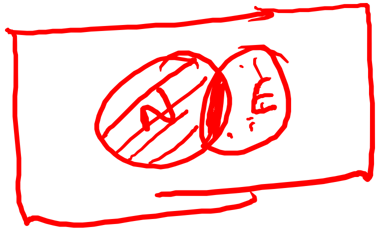
QUANTITATIVE TECHNIQUES FOR MANAGERIAL DECISION - 1 (QTMD1G21-1)



AXIOMS OF PROBABILITY

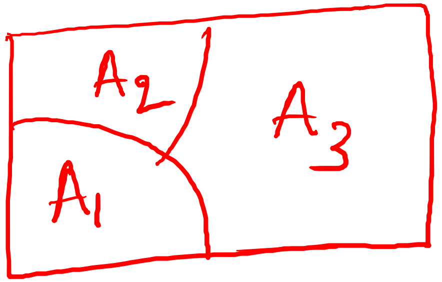
For any event A:

i) $0 \leq P(A) \leq 1$.

ii) $P(\Phi) = 0$, where Φ is the null event (no outcomes) and $\boxed{P(S) = 1}$, where S is the sample space (all outcomes).

iii) For any sequence of mutually exclusive events $A_1, A_2, \ldots$

$$P(A_1 U A_2 U \ldots) = P(A_1) + P(A_2) + \ldots$$



Tossing a coin and throwing a die.

$S = \{ (H,1), (H,2), (H,3), (H,4), \ldots, (T,1), (T,2), \ldots (T,6) \}$

A_1 A_2 A_3 A_7

$P($

A_3^c

$E_1 = \{1\}$, $E_1^c = \{2, \ldots 6\}$

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$,

$B = \{3, 6, 2\}$

$C = \{2, 4, 6\}$

PROBLEM

Consider the type of clothes dryer (gas or electric) purchased by each of five different customers at a certain store.

a) If the probability that at most one of these customers purchases an electric dryer is 0.428, what is the probability that at least two purchase an electric dryer?

$G_i E_{5-i}$

$S = \{G_5E_0, G_4E_1, G_3E_2, G_2E_3, G_1E_4, G_0E_5\}$

$A = \{G_5E_0, G_4E_1\}$, $A^c =$

$P(A) = 0.428$

$P(A^c) = 1 - 0.428 = 0.572$

PROBLEM

$S = \{C_{1G}, C_{2G}, C_{3G}, C_{4G}, \dots$

$S = \{1_G, 2_G, \dots\}$

Consider the type of clothes dryer (gas or electric) purchased by each of five different customers at a certain store.

b) If P(all five purchase gas) = 0.116 and P(all five purchase electric)=0.005, what is the probability that at least one of each type is purchased?

$= B$

$S = \{G_5E_0, G_4E_1, G_3E_2, G_2E_3, G_1E_4, G_0E_5\}$

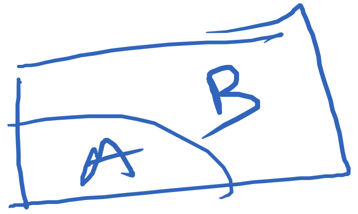
$B = \{G_5E_0\}$, $P(B) = 0.116$. $C = \{G_0E_5\}$, $P(C) = 0.005$

$1 - P(B) - P(C) = 0.879$

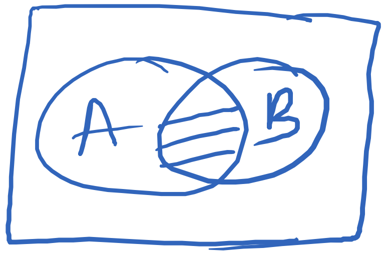
$S = \{(1,1), \dots (6,6)\}$

$S = \{(i,j) : 1 \le i \le 6, 1 \le j \le 6\}$

FURTHER RESULTS



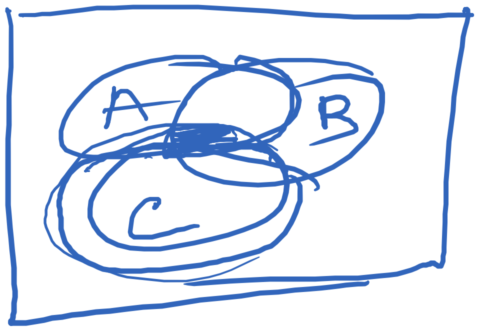
- If A and B are two mutually exclusive events such that P(AUB) = 1 then A and B are called complements of each other. We denote B as $\bar{A}$ or A^C.
- General formula for two events:



$$\underline{P(A \cup B) = P(A)+P(B)-P(A \cap B)}$$

- More than two events:

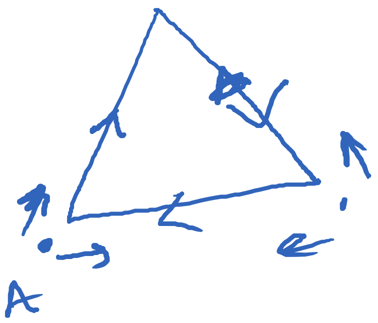
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)$
$\qquad - P(A \cap C) + P(A \cap B \cap C)$



PROBLEM 2.39, TEXTBOOK PG 51:

Three ants are sitting at the three corners of an equilateral triangle. Each ant starts to move randomly along an edge of the triangle. What is the probability that none of them collide with each other?

Not meet if ~~too~~ travelling clockwise or anticlockwise



Total No of choices for moving $= 2^3$

No. of favourable choices $= 2$

$\therefore P(\text{Not meeting}) = \frac{2}{2^3}$

$S = \{ A_{1c} A_{2A} A_{3A},\ A_{1c} A_{2c} A_{3A},\ \ldots \}$

PROBLEM 2.40, TEXTBOOK PG 51:

There are 3 computers available in an internet café. If on a particular day 3 customers arrived (at non-overlapping times), what is the probability that k computers ($k = 1, 2, 3$) were used on that particular day?

CONDITIONAL PROBABILITY AND INDEPENDENCE

A SIMPLE PROBLEM

$S = \{HH, HT, \cancel{TH}, TT\}$

Consider families with 2 children. Assume boys and girls are equally likely.

For the random experiment of choosing such a family, what would be the probability that one child is a girl?

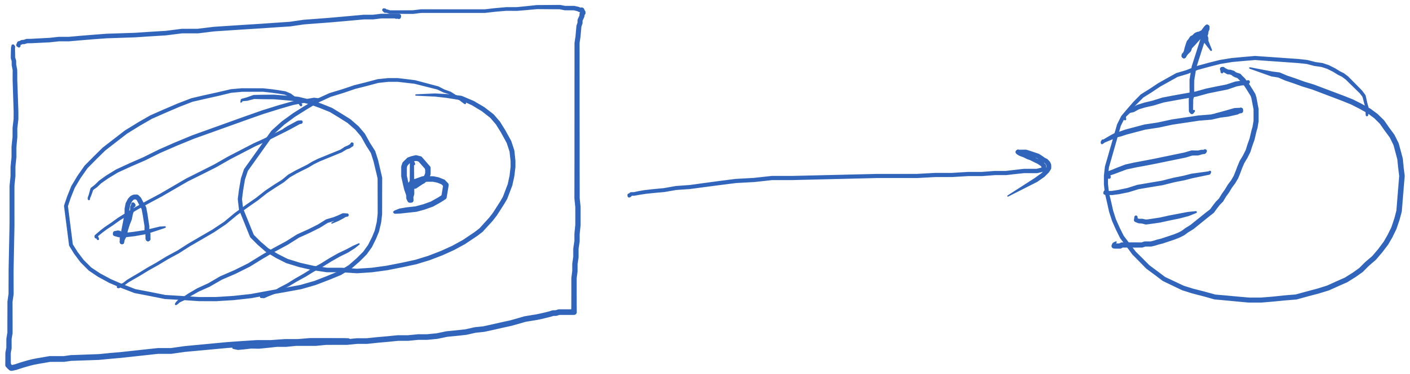
$S = \{BB, BG, GB, GG\}$

$P(\ ) = \frac{|E|}{|S|} = \frac{3}{4}$

$S^* = \{BG, GB, GG\}$

If one child of such a family is a girl, what is the probability that the other child is a girl?

$\frac{1}{3} = P(\ )$


A
B

CONDITIONAL PROBABILITY

Let A, B be two events such that P(A) >0. Then

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

is the conditional probability of B given A.

Conditional probability given an event A is computed by restricting the sample space to A.

P(B|A) and P(A∩B) are very different!

CONDITIONAL PROBABILITY

Conditional probability is also a probability: $Q(B) = P(B|A)$

Posterior Probability

- $0 \leq Q(B) \leq 1$
- $Q(\Phi) = 0$, $Q(A) = 1$.
- The summation law holds i.e. $P(Q_1 \cup Q_2 \cup \ldots) = P(Q_1) + P(Q_2) + \ldots$ if $B_1, B_2, \ldots$ are disjoint inside A

PROBLEM 3.2, TEXTBOOK PG 97:

A bag contains blue and red balls. Two balls are drawn randomly without replacement.

The probability of selecting a red and then a blue ball is 0.4. The probability of selecting a red ball in the first draw is 0.5.

What is the probability of drawing a blue ball, given that the first ball drawn was red?

Ans: $P(B \cap R) = 0.4$, $P(R) = 0.5$

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{0.4}{0.5} = 0.8$$

PROBLEM

An analyst estimates that the probability of default on a seven-year AA rated bond is 0.06, while that on a seven-year A rated bond is 0.13. If the seven-year AA rated bond defaults, then the chances of the seven-year A rated bond defaulting is 0.67.

a) What is the probability that both the bonds will default?

Ans: $D_{AA} \rightarrow$ AA bond, $D_A \rightarrow$ A bond,

$P(D_{AA}) = 0.06$, $P(D_A) = 0.13$

$P(D_A | D_{AA}) = 0.67$

$$P(D_A | D_{AA}) = \frac{P(D_A \cap D_{AA})}{P(D_{AA})}$$

$\boxed{P(D_A \cap D_{AA}) = ?}$

$\Rightarrow P(D_A \cap D_{AA}) = 0.0402$

PROBLEM

An analyst estimates that the probability of default on a seven-year AA rated bond is 0.06, while that on a seven-year A rated bond is 0.13. If the seven-year AA rated bond defaults, then the chances of the seven-year A rated bond defaulting is 0.67.

b) What is the probability that neither the seven-year AA rated bond nor the seven-year A rated bond will default?

$$P(D_{AA}^c \cap D_A^c) = 1 - P(D_{AA} \cup D_A) = 1 - P(D_{AA}) - P(D_A) + P(D_{AA} \cap D_A)$$

$$= 0.7698$$

MULTIPLICATIVE/CHAIN RULE

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

General Rule:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

SUB EVENTS AND SUPER EVENTS

A is a sub-event of B if whenever A happens, B must happen as well.

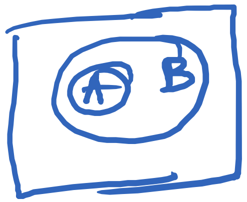
We write $A \subseteq B$ and call B a super-event of A.

$A = \{2, 4, 6\}$

$B = \{2\}$ $\quad B \subseteq A$

What happens to P(A|B) and P(B|A) when A is a sub-event of B?

$A \subseteq B$



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

INDEPENDENT EVENTS

Two events A and B are called independent if

$$P(A \cap B) = P(A)P(B)$$

Idea comes from the fact that A and B are independent means

$$P(A|B) = P(A|B^c) = P(A)$$

or equivalently $P(B|A) = P(B|A^c) = P(B)$.

Are mutually exclusive events independent events?

INDEPENDENT EVENTS

$$\left.\begin{array}{l} P(A\cap B) = P(A)P(B) \\ P(B\cap C) = P(B)P(C) \\ P(C\cap A) = P(C)P(A) \end{array}\right\}$$ Pairwise

For three events A, B and C, in addition to pairwise independence, we also need

$$P(A|B\cap C) = P(A|B^c\cap C) = P(A|B\cap C^c) = P(A|B^c\cap C^c) = P(A)$$

or a similar description for B or C.

Together with pairwise independence, all of the above conditions reduce to one single additional condition:

$$\boxed{P(A\cap B\cap C) = P(A)P(B)P(C)}$$

PROBLEM

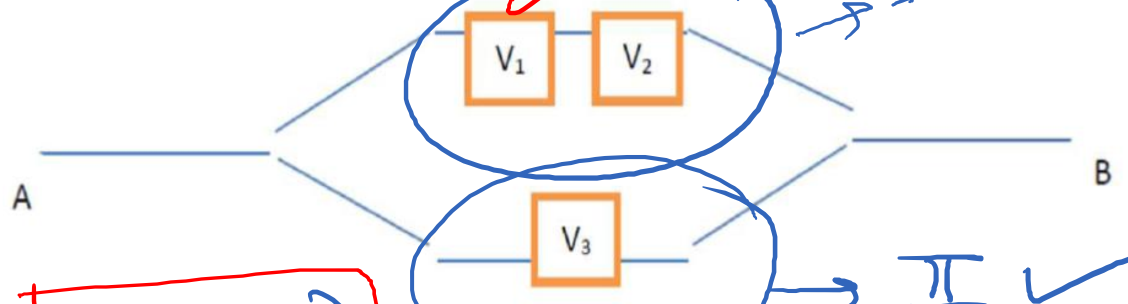
V_i: Working , V_i^c: Not working

$$\boxed{\begin{aligned} P(V_i^c) &= p \\ P(V_i) &= 1-p \end{aligned}}$$

$i = 1, 2, 3$

The valves in the image below develop faults and stop working independently with probability p each.

a) What is the probability that gas is flowing from A to B?



$\rightarrow$ I

$\rightarrow$ II

$P(\text{gas flows from A to B})$

$$= P(\mathrm{I} \cup \mathrm{II}) = P(\mathrm{I}) + P(\mathrm{II}) - \boxed{P(\mathrm{I} \cap \mathrm{II})}$$

$$P(\mathrm{II}) = P(V_3) = 1-p$$

$$P(\mathrm{I}) = P(V_1 \cap V_2) = P(V_1)P(V_2) = (1-p)^2$$

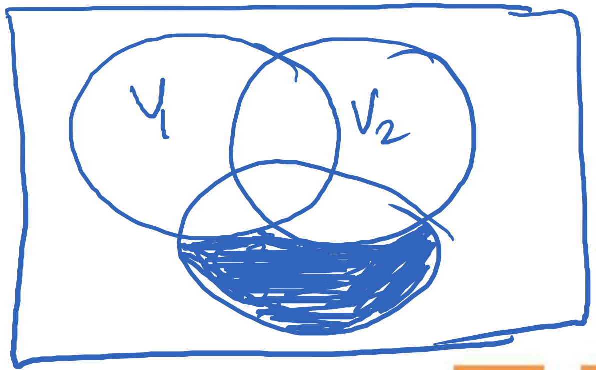
$$P(\mathrm{I} \cap \mathrm{II}) = P(V_1 \cap V_2 \cap V_3) = P(V_1)P(V_2)P(V_3)$$

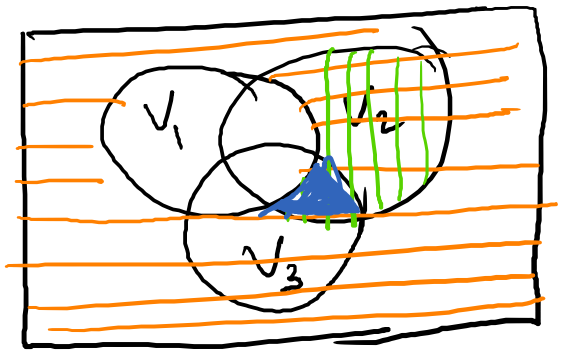
$$P(\mathrm{I} \cap \mathrm{II}) = P(V_1)P(V_2)P(V_3) = (1-p)^3$$

$$P(\text{gas flowing from } A \text{ to } B) = (1-p)^2 + (1-p) - (1-p)^3$$

$$= (1-p)(1+p-p^2)$$

G_{AB} : gas flowing from A to B.

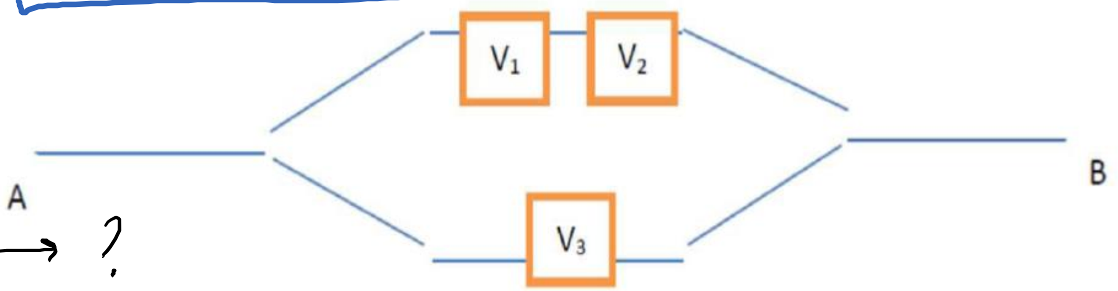




PROBLEM

The valves in the image below develop faults and stop working independently with probability p each.

b) Given that gas is flowing from A to B, what is the probability that V_1 has developed a fault?

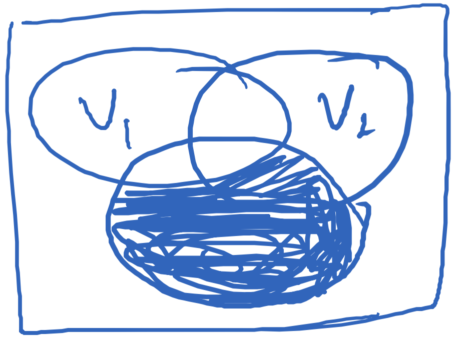


Ans: $P(V_1^c \mid G_{AB}) = \dfrac{\boxed{P(V_1^c \cap G_{AB})} \rightarrow ?}{\boxed{P(G_{AB})} \rightarrow \text{we know from part (a)}}$

$$P(V_1^c \cap G_{AB}) = \boxed{P(V_1^c \cap V_2 \cap V_3) + P(V_1^c \cap V_2^c \cap V_3)}$$

$$= P(V_1^c \cap V_3) = P(V_1^c)\,P(V_3)$$

$$= p(1-p)$$



$$P(V_i^c \mid G_{AB}) = \frac{P(V_i^c \cap G_{AB})}{P(G_{AB})} = \frac{p(1-p)}{(1-p)(1+p-p^2)}$$

$$= \frac{p}{1+p-p^2}$$

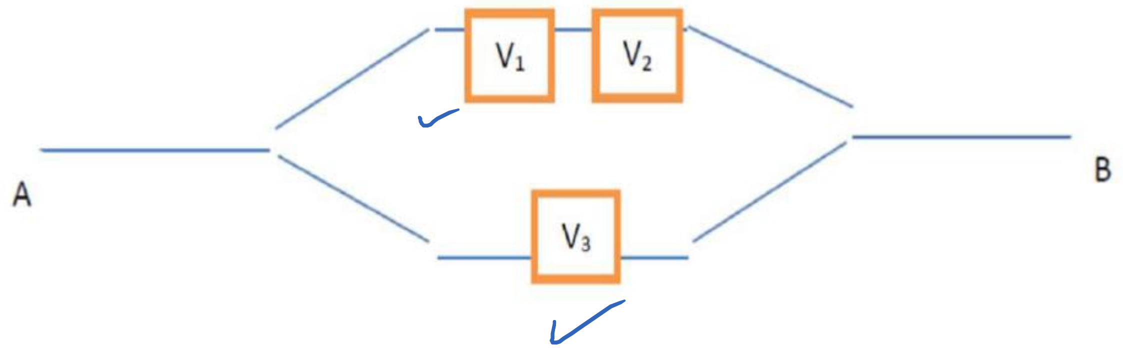
PROBLEM

$$P(V_1 | G_{AB}^c) = \frac{P(V_1 \cap G_{AB}^c)}{P(G_{AB}^c)}$$

$$= \frac{p^2(1-p)}{p^2(2-p)} = \frac{1-p}{2-p}$$

The valves in the image below develop faults and stop working independently with probability p each.

c) Given that gas is not flowing from A to B, what is the probability that V_1 has not developed a fault?



$$P(V_1 | G_{AB}^c) = \frac{P(V_1 \cap G_{AB}^c)}{P(G_{AB}^c)}$$

$$P(G_{AB}^c) = 1 - P(G_{AB}) = 1 - (1-p)(1+p-p^2) = p^2(2-p)$$

$$P(V_1 \cap G_{AB}^c) = P(V_1 \cap V_2^c \cap V_3^c) = P(V_1)\,P(V_2^c)\,P(V_3^c) = p^2(1-p)$$

TREE DIAGRAM

When events happen in sequence, it is often beneficial to attempt the problem using a probability tree.

PROBLEM 3.18, TEXTBOOK PG 98:

I : Ind, L : Lib

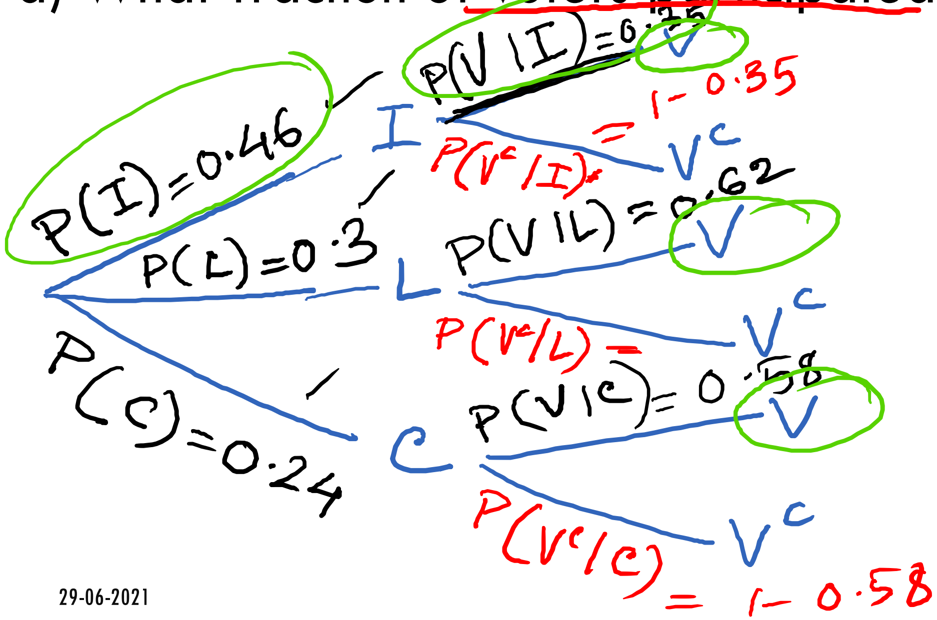
C : Conservative

V : Vote

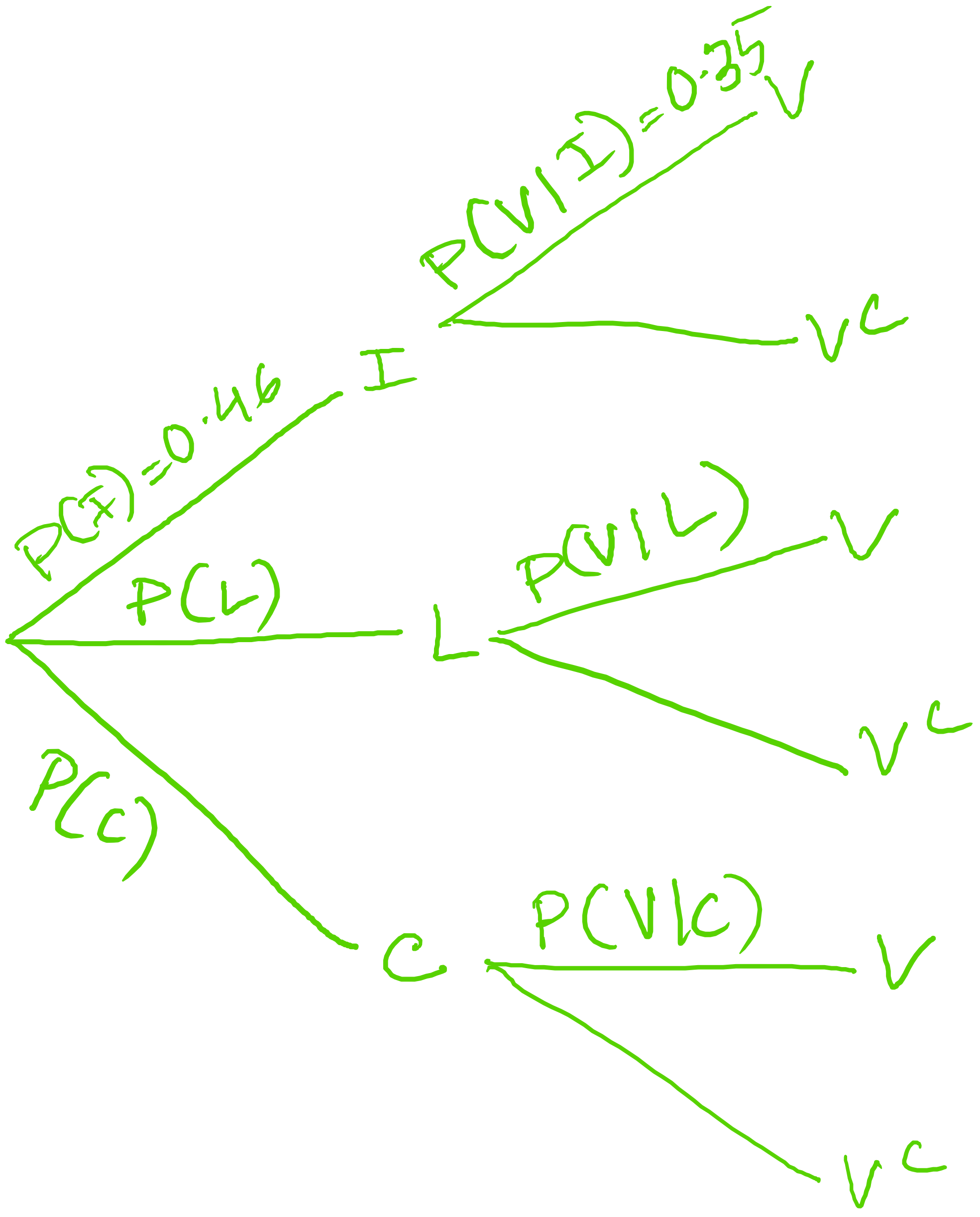
A total of 46% of the voters in a certain city classify themselves as Independents, whereas 30% classify themselves as Liberals and 24% say that they are Conservatives.

In a recent local election, 35% of the Independents, 62% of the Liberals, and 58% of the Conservatives voted. A voter is chosen at random.

a) What fraction of voters participated in the local election?



$$P(V) = P(V\cap I) + P(V\cap L) + P(V\cap C)$$
$$= P(I)P(V|I) + P(L)P(V|L) + P(C)P(V|C)$$
$$= 0.4862$$



Total Prob

$$P(V) = P(I)\,P(V|I) + P(L)\,P(V|L) + P(C)\,P(V|C)$$

$$\boxed{P(C|V) = \frac{P(C)\,P(V|C)}{P(I)\,P(V|I) + P(L)\,P(V|L) + P(C)\,P(V|C)}}$$

Bayes' ~~Road~~ Rule!

PROBLEM 3.18, TEXTBOOK PG 98:

A total of 46% of the voters in a certain city classify themselves as Independents, whereas 30% classify themselves as Liberals and 24% say that they are Conservatives.

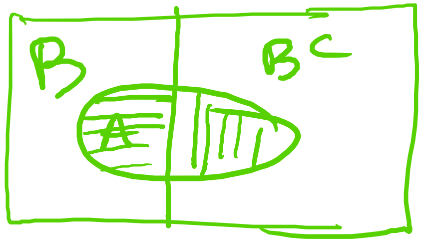
In a recent local election, 35% of the Independents, 62% of the Liberals, and 58% of the Conservatives voted. A voter is chosen at random.

b) Given that this person voted in the local election, what is the probability that he or she is a Conservative?

$$P(C \mid V) = \frac{P(C \cap V)}{P(V)} = \frac{0.24 \times 0.58}{0.4862} = 0.2863019$$

TOTAL PROBABILITY AND BAYES' THEOREM

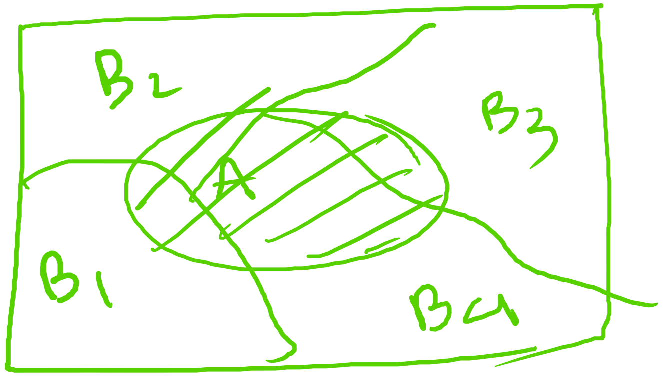
LAW OF TOTAL PROBABILITY



Let A and B be two events such that $0 < P(A) < 1$. Then

$$P(A) = P(B)P(A \mid B) + P(B^c)P(A \mid B^c)$$

$B_1, B_2, .. B_k \rightarrow$ Mutually exclusive and exhaustive events



$A \rightarrow$ any event

$$P(A) = \sum_{i=1}^{k} P(B_i) P(A \mid B_i)$$

JOINT AND MARGINAL PROBABILITY

The probability of two or more events occurring together or in succession is called the joint probability.

Marginal probability is the probability of the occurrence of the single event.

FALSE POSITIVE

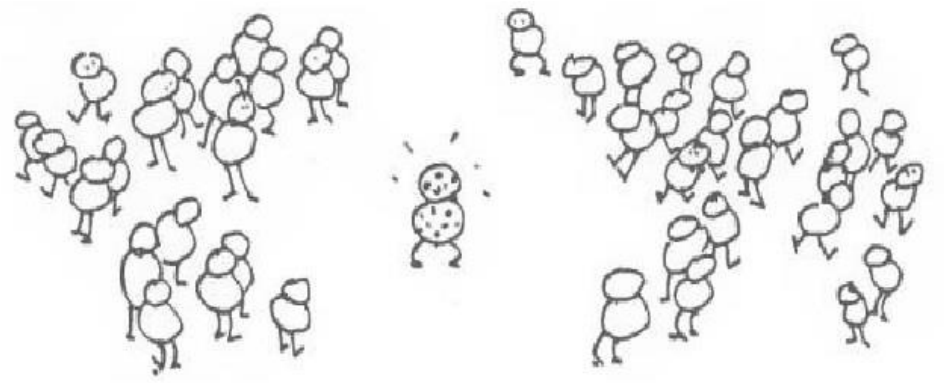
Suppose that a laboratory test on a blood sample yields one of two results, positive or negative.

It is found that 95% of people with a particular disease produce a positive result.

But 2% of the people without the disease will also produce a positive result (a false positive).

Suppose that 1% of the population actually has the disease.

What is the probability that a person chosen at random from the population would have the disease, given that the person's blood yields a positive result?



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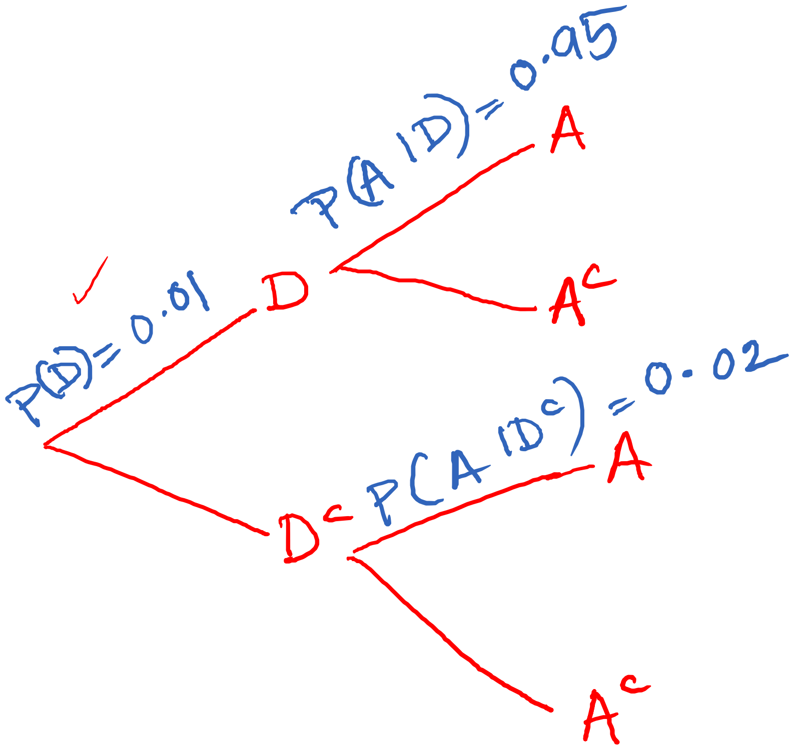
Suppose that 1% of the population actually has the disease.

What is the probability that a person chosen at random from the population would have the disease, given that the person's blood yields a positive result?

D : Person has disease

A : Test gives +ve result.



$$P(D|A) = \frac{P(D \cap A)}{P(A)}$$

$$= \frac{P(D)P(A|D)}{P(D)P(A|D) + P(D^c)P(A|D^c)}$$

$$= 0.324$$

FALSE POSITIVE

$\boxed{0.3\cancel{4}24}$

Suppose that a laboratory test on a blood sample yields one of two results, positive or negative.

It is found that $\boxed{95\%}$ of people with a particular disease produce a positive result.

But 2% of the people without the disease will also produce a positive result (a false positive).

Suppose that $\boxed{1\%}$ of the population actually has the disease.

What is the probability that a person chosen at random from the population would have the disease, given that the person's blood yields a positive result?

Sensitivity : $P(A \mid D) = 0.95$

Specificity : How specific positive results are to this disease

$P(A^c \mid D^c) = 0.98$

SOME NOTATIONS

A partition of the sample space: B_1, B_2, …, B_n

$P(B_i)$: "Prior Probability" of B_i

→ the unconditional probability of B_i

A: Some event

$P(B_i|A)$: "Posterior Probability" of B_i

→ Bayesian Update of probability of B_i

WE FOUND THAT $P(D \mid A) = 0.324$

$P(D) = 0.01$

The result seems counterintuitive!

The diagnostic test appears so accurate, we expect someone with a positive test result to be highly likely to have the disease, whereas the computed conditional probability is only 0.324!

However, because the disease is rare and the test isn't perfectly reliable, most positive test results arise from errors rather than from diseased individuals.

The probability of having the disease has increased by a multiplicative factor of 32.4 (from prior .01 to posterior 0.324);

But to get a further increase in the posterior probability, a diagnostic test with much smaller error rates is needed.

If the disease were not so rare (e.g., 25% incidence in the population), then the error rates for the present test would provide good diagnoses.

$P(D) = 0.25 \longrightarrow P(D \mid A) = 0.94$

BAYES' THEOREM

Let A and B be two events such that 0<P(B)<1 and P(A)>0. Then

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} = \frac{P(A|B)P(B)}{P(A)}$$

$B_1, B_2, \ldots B_k \rightarrow$ Mutually exclusive and exhaustive events.

$P(B_i) > 0, \quad i = 1, 2, \ldots, k.$

Then for event A, $(P(A) > 0)$,

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}, \quad j = 1, 2, \ldots, k.$$

CONTINGENCY TABLE

CUSTOMER SATISFACTION SURVEY FOR DIFFERENT FUNDS

An investment broker sells several kinds of investment products - an equity fund, a bond fund, and a money market fund.

The broker wishes to study whether client satisfaction with its products and services depends on the type of investment product purchased.

To do this, 100 of the broker's clients are randomly selected from the population of clients who have purchased shares in exactly one of the funds.

The broker records the fund type purchased by each client and has one of its investment counselors personally contact the client.

When contacted, the client is asked to rate his or her level of satisfaction with the purchased fund as high, medium, or low.

FundType	SatisfactionRating
BOND	HIGH
Equity	HIGH
MoneyMarket	MED
MoneyMarket	MED
Equity	LOW
Equity	HIGH
Equity	HIGH
BOND	MED
MoneyMarket	LOW
MoneyMarket	LOW
Equity	MED
BOND	LOW
Equity	HIGH
MoneyMarket	MED

CUSTOMER SATISFACTION SURVEY FOR DIFFERENT FUNDS

	H	M	L
Bond			
Equity			
MM			

FundType	SatisfactionRating
BOND	HIGH
Equity	HIGH
MoneyMarket	MED
MoneyMarket	MED
Equity	LOW
Equity	HIGH
Equity	HIGH
BOND	MED
MoneyMarket	LOW
MoneyMarket	LOW
Equity	MED
BOND	LOW
Equity	HIGH
MoneyMarket	MED

CUSTOMER SATISFACTION SURVEY FOR DIFFERENT FUNDS

	High	Low	Medium	Total
Bond	15	3	12	30
Equity	24	2	4	30
MoneyMarket	1	15	24	40
Total	40	20	40	100

Equity and Medium

Total no. of obs.

CUSTOMER SATISFACTION SURVEY FOR DIFFERENT FUNDS

$P(E \cap L) = 0.02$

	High	Low	Medium	Total
Bond	0.15	0.03	0.12	0.30
Equity	0.24	0.02	0.04	0.30
MoneyMarket	0.01	0.15	0.24	0.40
Total	0.40	0.20	0.40	1.00

CUSTOMER SATISFACTION SURVEY FOR DIFFERENT FUNDS

	High	Low	Medium	Total
Bond	**0.15**	**0.03**	**0.12**	**0.30**
Equity	**0.24**	**0.02** = P(E∩L)	**0.04**	**0.30**
MoneyMarket	**0.01**	**0.15**	**0.24**	**0.40**
Total	**0.40**	**0.20**	**0.40**	**1.00**

P(B)

Joint Probability

Marginal Probability

P(M) = 0.4

PROBLEM 3.21, TEXTBOOK PG 99:

A total of 500 married working couples were polled about their annual salaries, with the following information resulting:

Wife	Husband: Less than \$25,000		Husband: More than \$25,000			
Less than \$25,000	0·424	212	0·396	198	410	0·82
More than \$25,000	0·072	36	0·108	54	90	0·18
		248		252	500	
		0·494		0·504	1·0	

If one of the couples is randomly chosen, what is

(a) the probability that the husband earns less than \$25,000?

a) $P(H_{<25000})$

$= 0{\cdot}494$

PROBLEM 3.21, TEXTBOOK PG 99:

A total of 500 married working couples were polled about their annual salaries, with the following information resulting:

Wife	Husband: Less than \$25,000	Husband: More than \$25,000
Less than \$25,000	212	198
More than \$25,000	36	54

0.108

0.504

If one of the couples is randomly chosen, what is

(b) the conditional probability that the wife earns more than \$25,000 given that the husband earns more than this amount?

$$P(W_{>25000} \mid H_{>25000})$$

$$= \frac{P(W_{>25000} \cap H_{>25000})}{P(H_{>25000})}$$

$$= \frac{0.108}{0.504} = 0.214$$

PROBLEM 3.21, TEXTBOOK PG 99:

A total of 500 married working couples were polled about their annual salaries, with the following information resulting:

Wife	Husband: Less than \$25,000	Husband: More than \$25,000
Less than \$25,000	212	198
More than \$25,000	36	54

0.072

0.494

If one of the couples is randomly chosen, what is

(c) the conditional probability that the wife earns more than \$25,000 given that the husband earns less than this amount?

$$P(W_{>25000} \mid H_{<25000}) = \frac{P(W_{>25000} \cap H_{<25000})}{P(H_{<25000})}$$

$$= \frac{0.072}{0.494} = 0.145$$

ODDS OF AN EVENT

The odds in the favor of an event A is the ratio of the probability of A to the probability of A^C i.e.

$$\text{Odds in favor of A} = \frac{P(A)}{P(A^c)}$$