QUANTITATIVE TECHNIQUES FOR MANAGERIAL DECISION - 1 (QTMD1G21-1)

Geometric (p)

$P(X=k) \longrightarrow$ R p.m.f. : dgeom (x, p)

prob of success

↓ # failures

at k-th trial the 1^{st} success appears

p.m.f.

$P(X=k) \rightarrow 1^{st}$ success after $k-1$ failures

$= \boxed{P(\#f = k-1)}$

[$\#f$ = No of failures]

cdf : $P(X \leq k)$

pgeom $(x, p) \longrightarrow$ prob of success

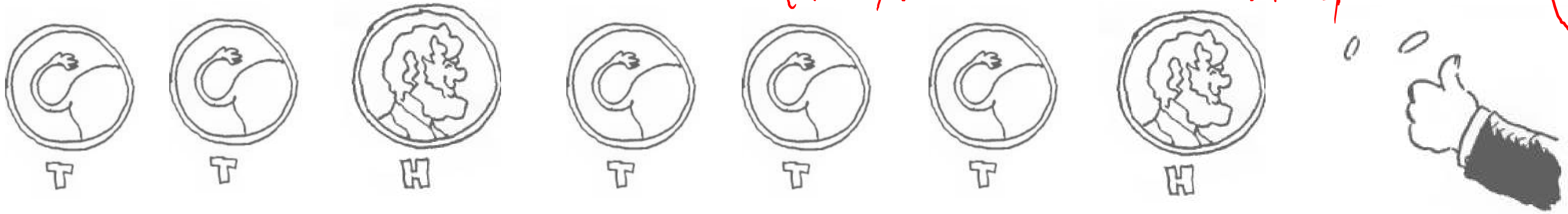
↓ # of failures

NEGATIVE BINOMIAL RANDOM VARIABLE

The experiment continues (trials are performed) until a total of r successes have been observed,

where r is a specified positive integer.

T T T H T H T T T H $\binom{10-1}{3-1}(1-p)^{10-3}p^3$



X = Number of trials needed to obtain the r-th success

$$P(X = x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$$

E(X) = r/p,

Var(X) = r(1-p)/p^2

NEGATIVE BINOMIAL DISTRIBUTION: GENERAL SETUP

X: number of trials needed for r^{th} success

X ~ Negative Binomial(r,p)

(Note: no. of failures = no. of trials – r)

R:

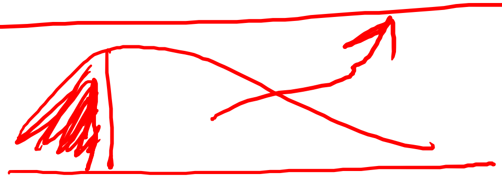
- p.m.f. of no. of failures: `dnbinom(x,r,p)`
- c.d.f. of no. of failures: `pnbinom(x,r,p)`

Survival function: $1 - F(x) = 1 - P(X \le x)$

`pnbinom(x,r,p,lower.tail = FALSE)`

$P(X=x)$

$P(X \le x) = F(x)$



ADDITIVE PROPERTY

If $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(p)$ are independent, then $X+Y \sim NB(2,p)$.

In general, if $X \sim NB(r_1,p)$ and $Y \sim NB(r_2,p)$ are independent, then $X+Y \sim NB(r_1+r_2,p)$.

Note: p has to be the same, and independence is needed.

PROBLEM

Suppose that a telemarketing company salesman is making phone calls to find potential customers. She is new to her job. Her seniors in the company told her that the chances of making a sale is 2% initially.

a) What is the probability that she makes no sale in her first 10 phone calls?

$P(\text{Sale}) = 0.02$

X = # of phone calls to get the first sale

$\boxed{P(X > 10)} = 1 - P(X \leq 10) = 1 - \text{pgeom}(9, 0.02)$

$= \boxed{0.8170728} \simeq 0.8171$

$P(X = 10) \rightarrow \text{pmf}$

PROBLEM

$P(X = x) \rightarrow$ dgeom

$P(X \leq x) \rightarrow$ pgeom

Suppose that a telemarketing company salesman is making phone calls to find potential customers. She is new to her job. Her seniors in the company told her that the chances of making a sale is 2% initially.

b) What is the probability she makes her first sale in the 11th phone call?

$$P(X=11) = (0.98)^{10} \times 0.02 = \text{dgeom}(10, 0.02)$$

$$= 0.01634$$

PROBLEM

1
2
3
⋮

11

Suppose that a telemarketing company salesman is making phone calls to find potential customers. She is new to her job. Her seniors in the company told her that the chances of making a sale is 2% initially.

c) What is the probability that there is at least one sale in the first 11 phone calls?

$$P(X \leq 11) = \text{pgeom}(10, 0.02)$$

$$= 0.199$$

PROBLEM

$P(X=x)$: dnbinom(#f, r, p) → prob of success; #f = No. of failures; r = No of success

$$P(X=x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

Suppose that a telemarketing company salesman is making phone calls to find potential customers. She is new to her job. Her seniors in the company told her that the chances of making a sale is 2% initially.

d) Suppose that her manager has given her a target. She must make 5 sales by the end of the week. What is the probability that she can meet her target on 50th phone calls?

X = # of successes in 50 phonecalls

$X \sim NB(5, 50, 0.02)$

$$P(X=5) = \boxed{\text{dnbinom}(45, 5, 0.02)} \rightarrow \text{R Code}$$

$$= \binom{49}{4}(0.02)^5(0.98)^{45} = 0.00027315 25$$

PROBLEM

$P(X=x) \rightarrow dnbinom(\underline{\#f, s, p})$

$P(X \le x) \rightarrow pnbinom(\quad)$

Suppose that a telemarketing company salesman is making phone calls to find potential customers. She is new to her job. Her seniors in the company told her that the chances of making a sale is 2% initially.

e) What is the probability that she makes at least 3 sales in the first 50 phone calls?

$P(X \le 50)$

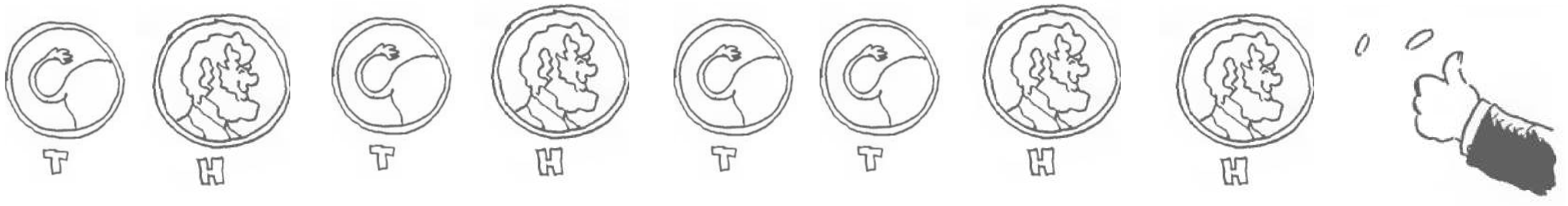
$= P(\#f \le 47)$

$= pnbinom(47, 3, 0.02)$

$= 0.07842775$

MORE WITH BERNOULLI TRIALS…

Let us keep on repeating the Bernoulli Trials…



… and count the number of time we get “success”.

X_1
X_2
$\vdots$
X_n

$Var(Y) = Var(X_1 + \cdots + X_n)$
$= Var(X_1) + \cdots + Var(X_n)$
$= np(1-p)$

$Y = X_1 + X_2 + \cdots + X_n$
$\quad 1,0 \quad 1,0 \quad 1,0$

BINOMIAL RANDOM VARIABLE: BIN(N,P)

n independent trials with probability of success p

X = Number of times “success” occurs

A binomial random variable X is defined as the number of successes achieved in the n trials of a Bernoulli process.

Notation: $X \sim Bin(n, p)$

p.m.f.: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \ldots, n.$
$E(X) = np$
$Var(X) = np(1-p)$
$SD(X) = \sqrt{np(1-p)}$

In R:

- p.m.f. of the number of successes: `dbinom(x,n,p)`
- c.d.f. of the number of successes: `pbinom(x,n,p)`

No. of success
No. of trials
Prob of success.

$E(Y)$
$= E(X_1 + X_2 + \cdots + X_n)$
$= E(X_1) + E(X_2) + \cdots + E(X_n)$
$= p + p + \cdots + p$
$= np$

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

PROBLEM 4.25, TEXTBOOK PG. 165

$= 12$

A box containing a dozen eggs are examined. The probability that an egg would be broken is 0.1 and is independent of whether other eggs are broken or not.

If X denotes the number of broken eggs in the box,

a) Find P(X=0)

X = # of broken eggs in a box.

$$X \sim Bin(\underset{n}{12}, \underset{p}{0.1})$$

$$P(X=0) = \binom{12}{0} (0.1)^0 (0.9)^{12} = dbinom(0,12,0.1)$$

$$= 0.2824295$$

PROBLEM 4.25, TEXTBOOK PG. 165

A box containing a dozen eggs are examined. The probability that an egg would be broken is 0.1 and is independent of whether other eggs are broken or not.

If X denotes the number of broken eggs in the box,

b) If a box containing more than 1 broken egg is rejected, calculate the probability that a given box will be rejected?

Box rejected if $P(X > 1) = P(X \geq 2) = P(X=2) + P(X=3) + \cdots + P(X=12)$

$= 1 - P(X \leq 1)$

$= 1 - \text{pbinom}(1, 12, 0.1)$

$= 0.34099977$

$\simeq 0.341$

PROBLEM 4.25, TEXTBOOK PG. 165

A box containing a dozen eggs are examined. The probability that an egg would be broken is 0.1 and is independent of whether other eggs are broken or not.

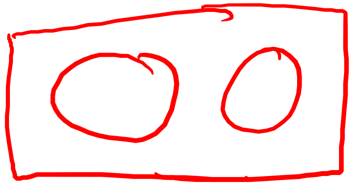
If X denotes the number of broken eggs in the box,

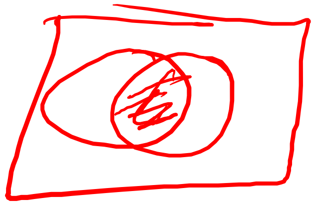
c) Find E(X).

$$E(X) = np = 12 \times 0.1 = 1.2$$

Ind: $X \sim Bin(10, \frac{1}{2})$, $Y \sim Bin(5, \frac{1}{2})$

$X+Y \sim Bin(15, \frac{1}{2})$



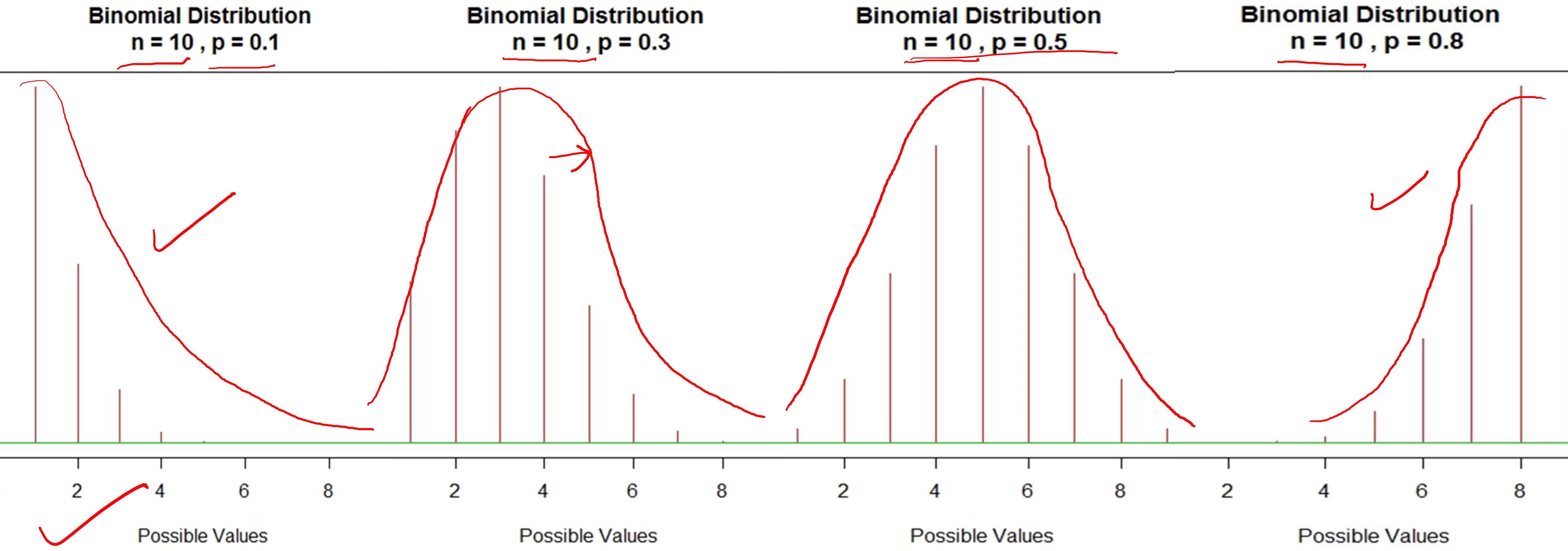


ADDITIVE PROPERTY

If X~Bin(n, p) and Y~Bin(m, p) are independent, then X+Y~Bin(n+m, p).

Note:

- p has to be the same, and independence is needed.
- The above also means that Bin(n,p) is sum of n independent Bin(1,p), or Ber(p).

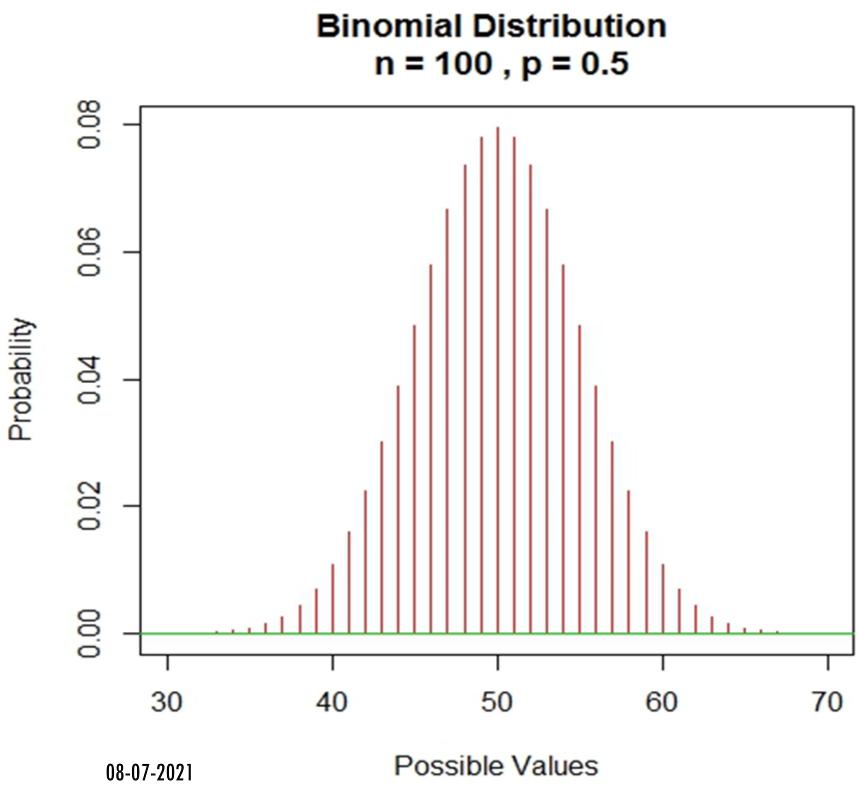


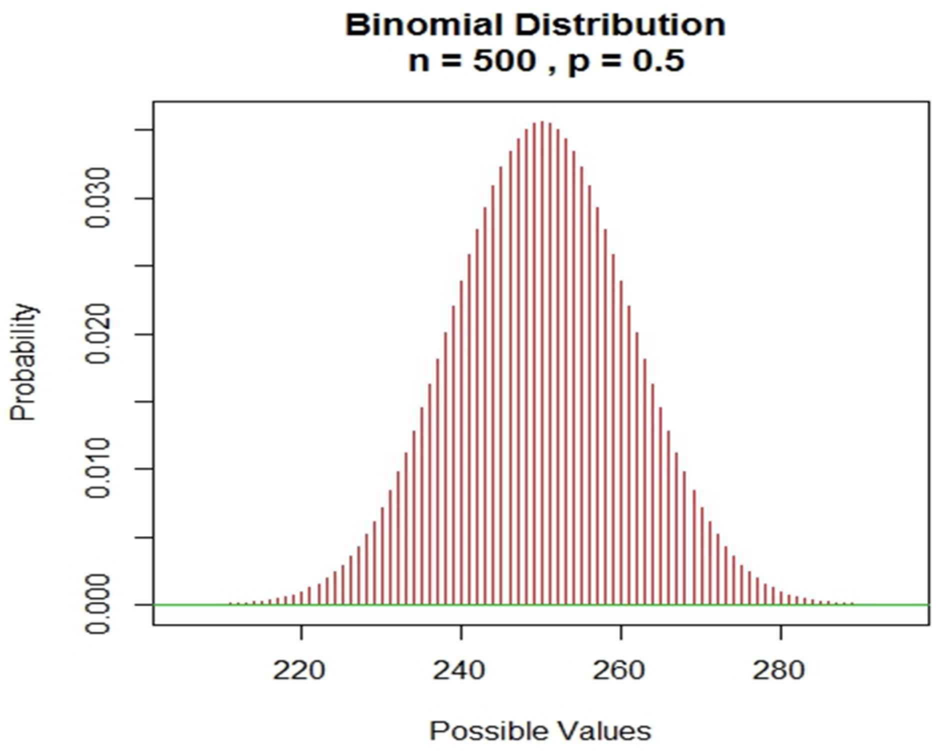
BINOMIAL DISTRIBUTION PLOTS

APPROXIMATIONS TO BINOMIAL DISTRIBUTION

Case 1. "Large n", np and n(1-p) "large". Use "normal" approximation.

We shall come back to this after we discuss continuous distributions.





APPROXIMATIONS TO BINOMIAL DISTRIBUTION

Case 2. "Large n", "small p",(or "small 1-p"), "moderate $\lambda = np$". Use "Poisson" approximation.

→ "Small p": rare event.

POISSON DISTRIBUTION

Used to model rare events

Also used to approximate a Bin(n,p) random variable with large n and small p such than np = λ is moderate

X = number of events: rate λ

p.m.f.: $P(X=x) = e^{-\lambda}\frac{\lambda^{x}}{x!}$; x = 0, 1, 2, …

E(X) = λ

Var(X) = λ

R:

- `dpois(x,λ)`
- `ppois(x,λ)`

PROBLEM 4.60, TEXTBOOK PG. 167

$ppois(x, \lambda)$
events Rate

In an express highway toll-booth, on average 120 cars pass a specified point in any given hour. It is estimated if more than 3 cars arrive at any given minute, there will be a congestion at the toll booth.

a) Find the probability that the toll booth will be congested at some specific minute.

In 60 minutes → 120 cars.

$\lambda_{hr} = 120$ cars $\lambda_{min} = 2$

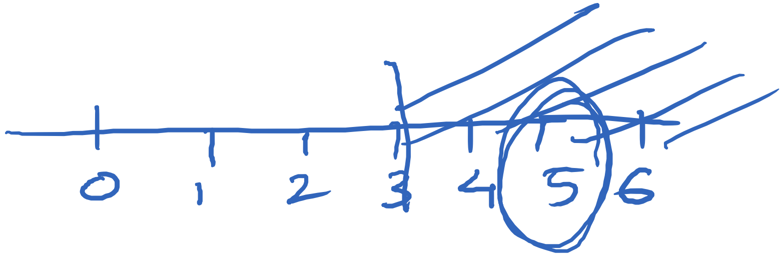
X = # of cars arriving at a given min

$$P(X > 3) = P(X \geq 4) = 1 - P(X \leq 3)$$
$$= (1 - \boxed{P(X=0) - P(X=1) - P(X=2)} - P(X=3))$$
$$= 1 - ppois(3, 2) = 0.1428765$$

PROBLEM 4.60, TEXTBOOK PG. 167

In an express highway toll-booth, on average 120 cars pass a specified point in any given hour. It is estimated if more than 3 cars arrive at any given minute, there will be a congestion at the toll booth.

b) Suppose you know that the toll-booth is congested between 11:05AM-11:06AM. What is the probability that there are only 5 cars in the toll-booth?



$$P(X=5 \mid X>3) = \frac{P(X=5 \cap X>3)}{P(X>3)}$$

$$= \frac{P(X=5)}{1-\text{ppois}(3,2)}$$

$P(X=5) \longrightarrow$ pmf $\longrightarrow$ dpois

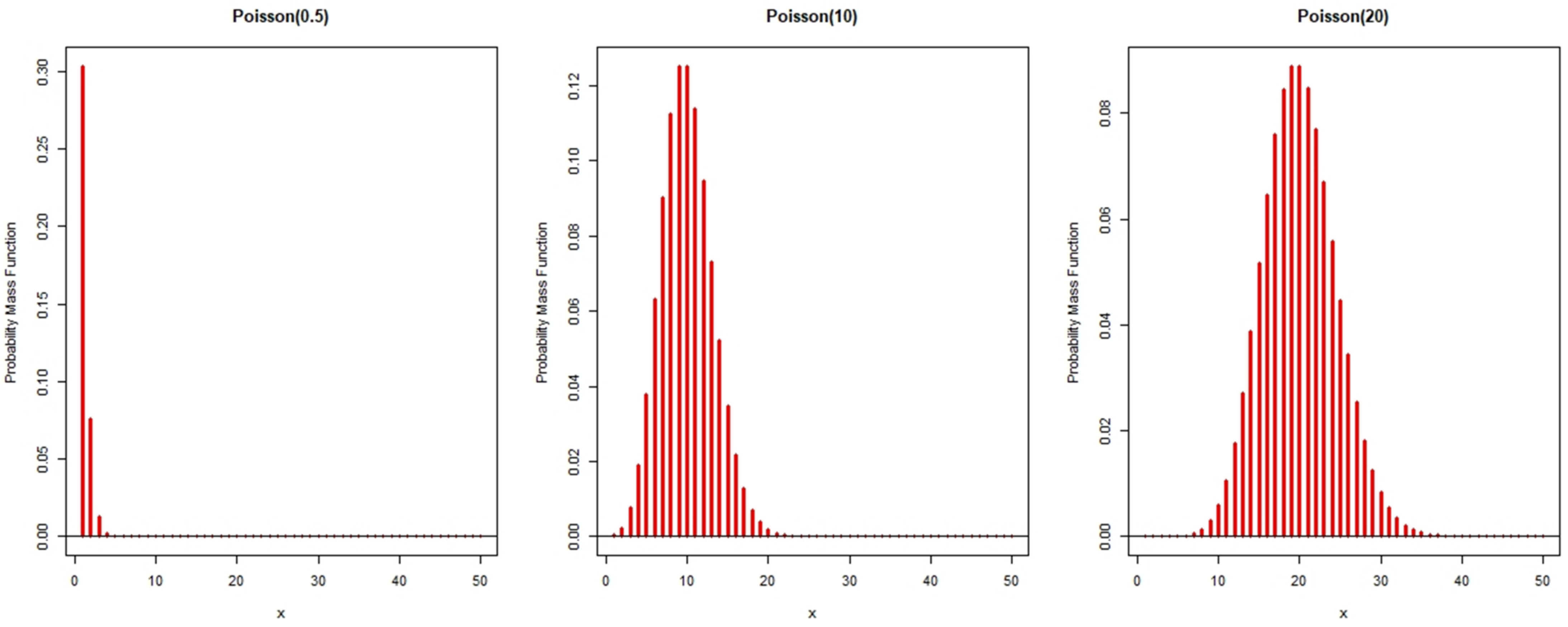
$$= \frac{\text{dpois}(5,2)}{1-\text{ppois}(3,2)} = 0.2525916$$

ADDITIVE PROPERTY

If $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ are independent, then $X+Y \sim \text{Poisson}(\lambda_1+\lambda_2)$.

Note:

- Independence is needed
- No restriction on rates


Poisson(0.5)
Poisson(10)
Poisson(20)
Probability Mass Function
x
0 10 20 30 40 50
0.00 0.05 0.10 0.15 0.20 0.25 0.30
0.00 0.02 0.04 0.06 0.08 0.10 0.12
0.00 0.02 0.04 0.06 0.08

SOME PROBLEMS

PROBLEM 4.40, TEXTBOOK PG. 166

$$\boxed{P(X=4) = \binom{10}{4}(0.3)^4(0.7)^6} = \text{dbinom}$$

A department of 10 employees has a wifi facility which connects a maximum of 3 devices at a time. If there is a 30% chance that an employee will need internet at a particular time, find the probability that wifi is not enough at a certain time.

Ans: X = # of employees using WIFI, $X \sim Bin(10, 0.3)$

$$P(X>3) = P(X \geq 4) = 1 - P(X \leq 3)$$

$$= 1 - \text{pbinom}(3, 10, 0.3)$$

$$= 0.3503893$$

$$\boxed{E(X) = np} = 10 \times 0.3 = 3$$

Problem 4.13, Textbook pg. 163

An airline operates a flight having 50 seats. As they expect some passengers to not show up, they overbook the flight by selling 51 tickets. The probability that an individual passenger will not show up is 0.01, independent of all other passengers.

Each ticket costs Rs. 10,000 and is non-refundable if a passenger fails to show up.

If a passenger shows up and a seat is not available, the airline has to pay a compensation of Rs. 1,00,000 to the passenger.

What is the expected revenue of the airline?

50 seats 51 ~~ti~~ sold.

X = # of passengers showing up.

$X \sim \text{Bin}(51, 0.99)$

~~Refund~~ Comp. not paid if $\underline{X \le 50}$ → earning Rs. (51×10000)

= Rs. 510000

Comp paid if $\underline{X = 51}$ → earning Rs. $(51 \times 10000 - 100000)$

= 410000

Earning	Prob
510000	$\underline{P(X \le 50)}$ = pbinom(50, 51, 0.99)
$\underline{41000}$	$\underline{P(X = 51)}$ = $\underline{\text{dbinom}(51, 51, 0.99)}$ =

$\binom{51}{51} \times (0.99)^{51} \times (0.01)^{0}$

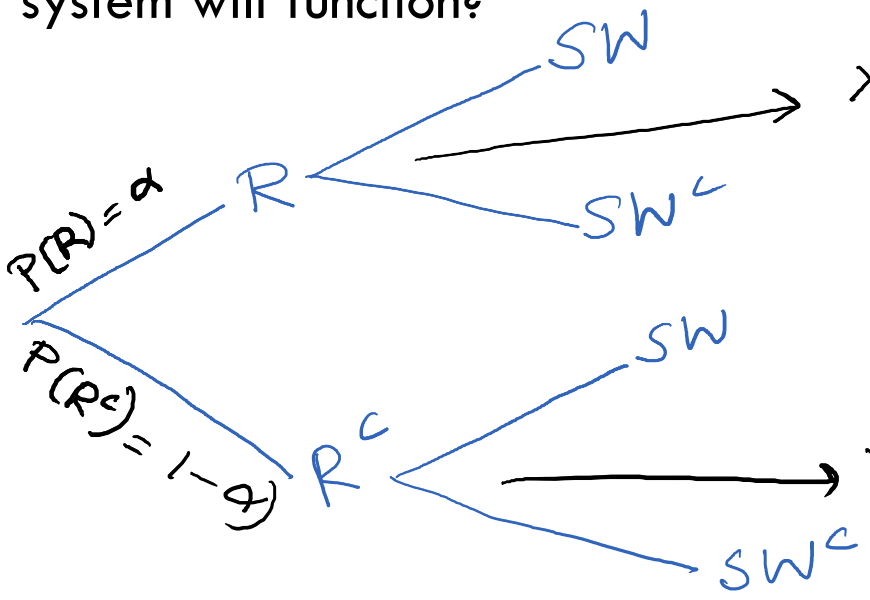
E(Revenue) = 510000 × pbinom(50, 51, 0.99)
+ 410000 × dbinom(51, 51, 0.99)

= 450104.2

R: Rainy day
SW: Satellite will work

PROBLEM 4.44, TEXTBOOK PG. 166

A satellite system consists of n components and functions on any given day if at least k of the n-components function on that day. On a rainy day, each of the components independently function with probability p_1, whereas on a dry day, each components independently function with probability p_2. If the probability of rain tomorrow is α, what is the probability that the satellite system will function?



X = # of components working on a rainy day

Then $X \sim Bin(n, p_1)$

$$P(SW|R) = P(X \geq k) = 1 - P(X < k)$$
$$= 1 - P(X \leq k-1)$$

Y = # of components working on a dry day

Then $Y \sim Bin(n, p_2)$

Then $Y \sim Bin(n, p_2)$

Then $P(SW|R^c) = P(Y \geq k) = 1 - P(Y \leq k-1)$

$\therefore P(SW) = \alpha P(X \geq k) + (1-\alpha) P(Y \geq k)$

$= \alpha \left[1 - P(X \leq k-1)\right] + (1-\alpha)\left[1 - P(Y \leq k-1)\right]$

2A 15

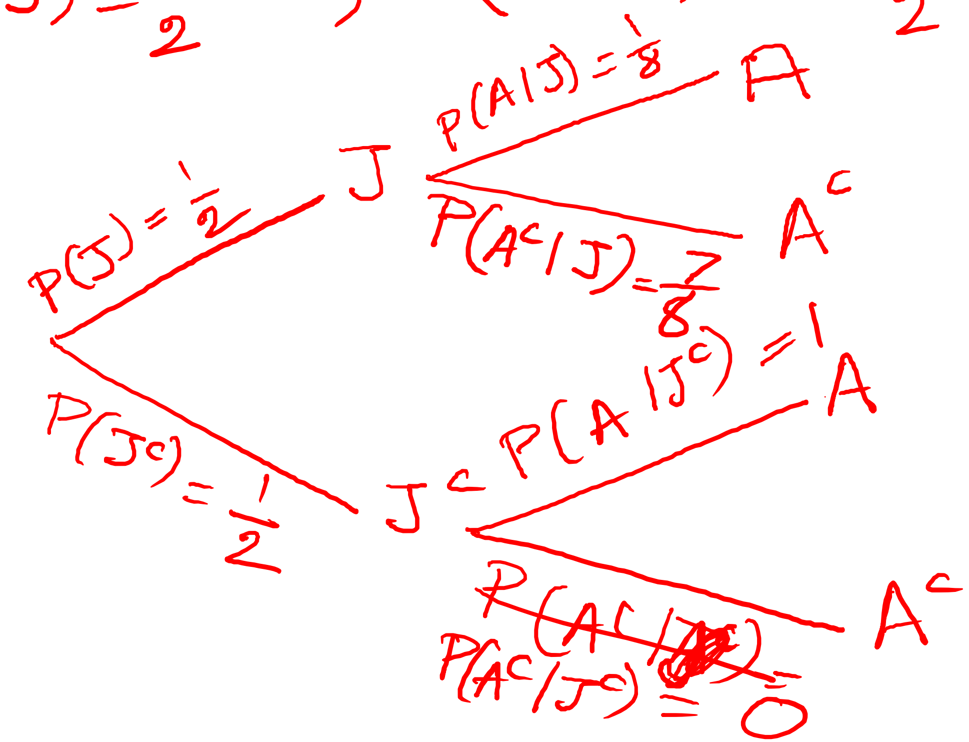
50 → 2B 48

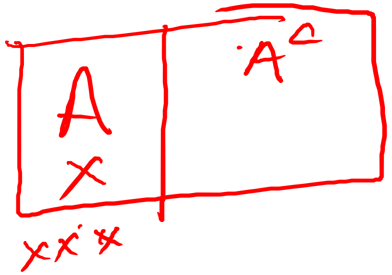
2B 2

J : Jane has disease , A : ~~None of~~ the children the disease.

A^c → at least one child has it.

$P(J) = \frac{1}{2}$, $P(A|J) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$





$$P(J|A) = \frac{P(J)\,P(A|J)}{P(J)\,P(A|J) + P(J^c)\,P(A|J^c)}$$

$$= 0.111$$