1 Intro to Binomial Model for Valuing Options

- Introduction to Binomial Model
  - One-step Binomial Model
  - Risk-Neutral Valuation
  - Two-step Binomial Trees
  - American Options
  - Delta
  - Multi-step Binomial Trees
  - Dividends (see sections 13.3 and 16.2)
  - Binomial Trees in Practice

- Recall: Put-Call parity with dividends $c - p = S_0 - Ke^{-rT} - D$

- See also http://www.cboe.com for calculator, IV and trading tools.

2 Intro to Binomial Model

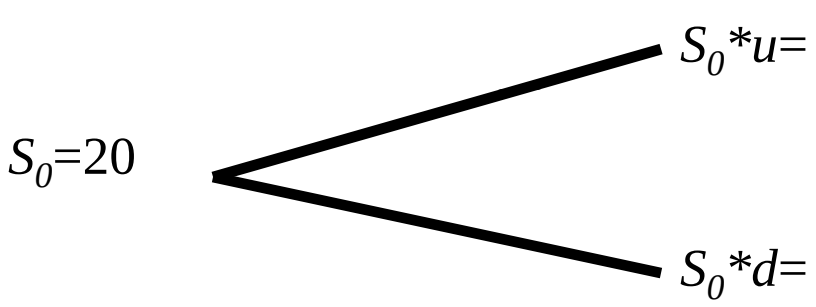
- Pricing options with “binomial model”
  - Binomial model based on *replication* and *arbitrage*.

- Basic “binomial model”
  - assume two periods and two possible values for stock price.
  - construct portfolio (long stock & short calls) that is perfectly hedged.
  - *riskless hedge portfolio* must earn the risk-less rate. Find implied call price.
  - Price put options from put-call parity $c - p = S_0 - Ke^{-rT} - D$

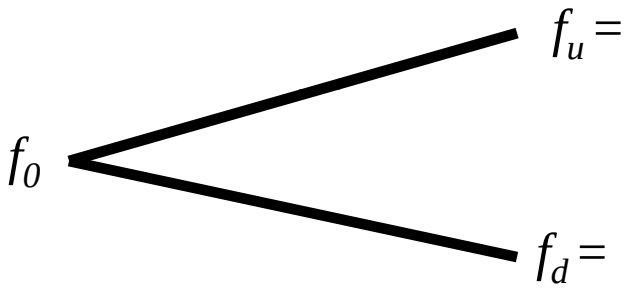
- Notes
  - Return from up/down movements must exceed riskless rate $d < e^{r(\delta t)} < u$
  - That is, expected return from risky investment must exceed riskless rate.

3 Binomial Model 1

- Step 1. Consider a stock (S_0=20) and a call option (K=\$21).
  - Given payoffs for stock (assume u=1.1; d=0.9), find payoffs for call option.
  - Note: binomial payoffs to stock and long call have Δ=1 (for short call Δ=–1).



Stock



Call Option K=21; T=3 mo.

- Step 2: Construct portfolio that *replicates* a risk-free investment. (next slide)
- Step 3: Price call option by equating return from *riskless hedge portfolio* (RHP) with risk-free rate. (next slide)

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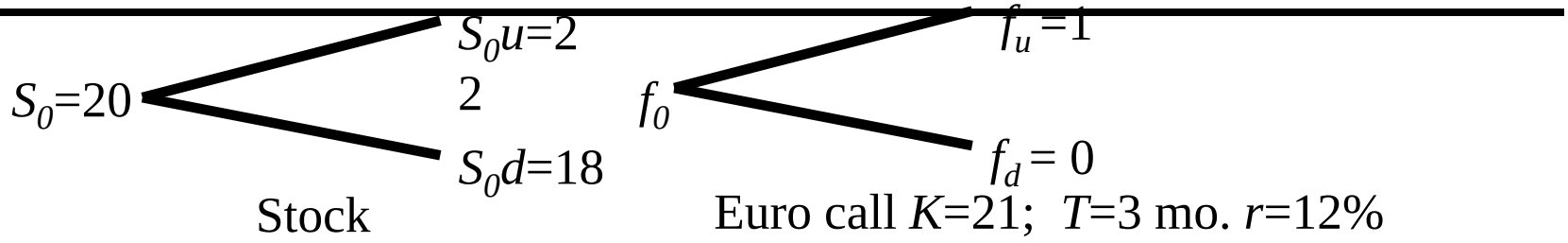
Binomial Model 2



- Step 2: Construct portfolio that *replicates* a risk-free investment (*RHP*).
  - Find *RHP* (long "👎" shares and short "1" call) with same payoff in each state.

    =

    👎 =

  - *RHP* Payoff: =

    =

  - *RHP* is long 0.25 shares per call option written w/ payoff=$4.50.

- Step 3: Price call by setting return on *RHP* equal to riskless rate

  Payoff/Cost of *RHP* = 1 + riskless rate

  Payoff/(👎$S_0 - f$) = $e^{r(\delta t)}$

5 Binomial Model 3

- If call option is, for example, over-priced then
  - *riskless hedge portfolio* will earn more than the risk-free rate.
  -

- Note on constructing the *riskless hedge portfolio*

  , the ratio of shares to options, is called the *hedge ratio.*

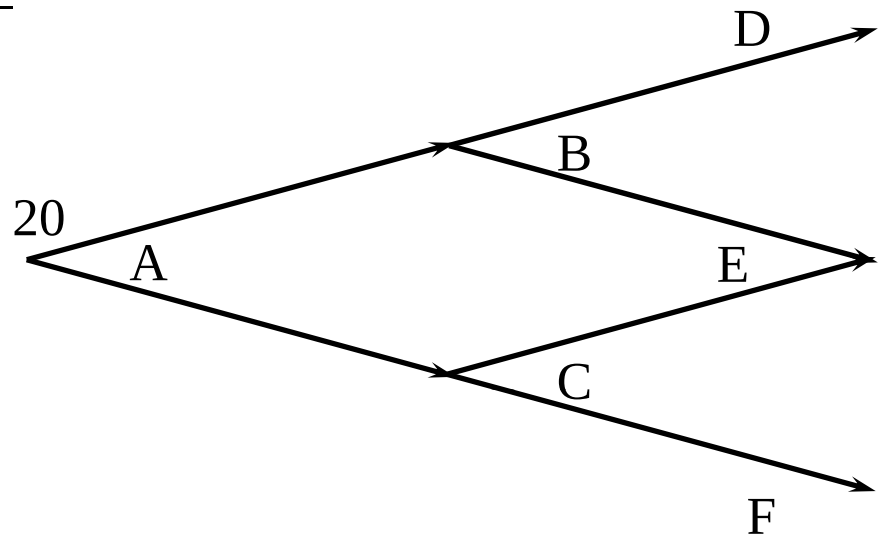
  For call options, if we know to go long stock and short calls,
  then hedge ratio for calls equals ratio of volatilities, $ = (f_u – f_d)/(S_0u – S_0d)$.
  where $u>1$ and $d<1$ (proportional change).

  Note: this comes from $S_0u* – f_u = S_0d * – f_d$

  Process is called *delta hedging*.

  of calls is positive; of puts is negative; varies from node to node.

6 Binomial Model with Two Steps



European call K=21;
T=6-mo; r=12%

- Binomial model with two-steps – start at end nodes and work backwards.
- Step 1: Consider a stock (S_0=??) and call option (K=??)
  - Compute payoff for call option at each terminal node, then work backward.
- Step 2: Construct *RHP* of stock and call that *replicates* a risk-free investment.
  - *Hedge ratio* for *RHP* at each node is $\Delta = (f_u - f_d) / (S_0u - S_od)$.
- Step 3: Price call at each node by setting *return on RHP equal to riskless rate*.

$$\Delta S_0 - f = (\Delta S_0 u - f_u) * e^{-r(\delta t)} \quad OR \quad (\Delta S_0 d - f_d) * e^{-r(\delta t)}$$

- Node B: Step 1: done; Step 2: Δ=0.7273; payoff=14.4 Step 3: f =
- Node C: Step 1: done; Step 2: Δ=0.0 Step 3: f =
- Node A: Step 1: done; Step 2: Δ=0.5075; payoff=9.14 Step 3: f =

7 Examples

- Example – suppose a stock currently trades for $50, and will be worth either $53 or $48 in two months. The risk free rate of interest is 10%. What is value of call option with an exercise of $49? Find hedge ratio, payoff to RHP, option price.
- A1:
- A2:
- A3

- Now find the price of the put option with the same characteristics. Note that you will be buying puts (not writing to form RHP).
- A1:
- A2
- A3: $f =$

- **Note: Expected Value and Pricing.** Suppose stock pays off either $40 or $60 in one year, how much would you pay today?
- What if payoff is $50 guaranteed?
- What if payoff is $0 or $100? What if payoff is -$50 or +$50?
- A:

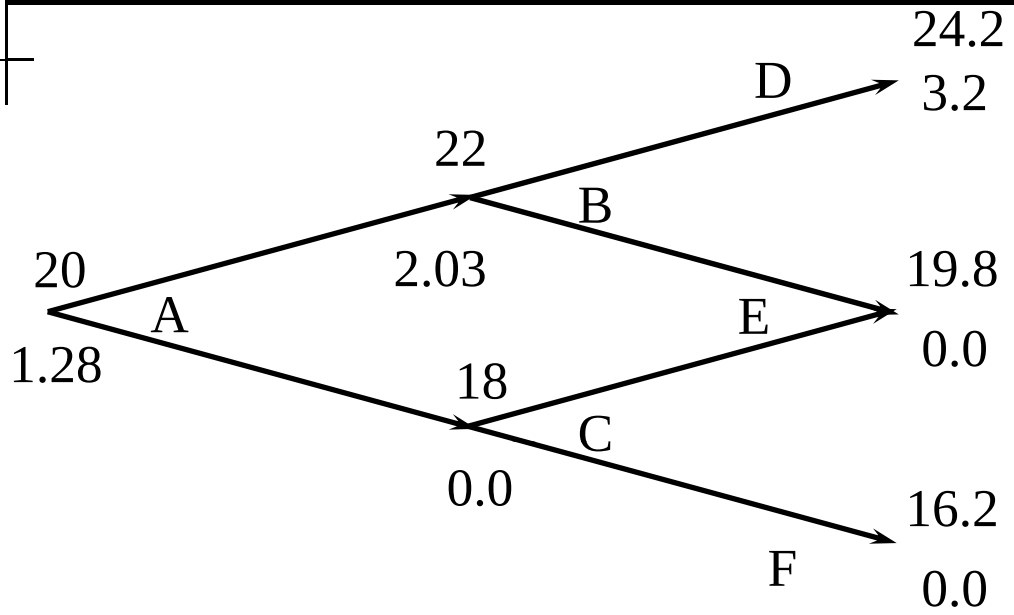
8 Binomial Model and Risk Neutral Valuation

- **Binomial Model** – note: *RHP* abbreviates *riskless hedge portfolio*
  - Step 1: Consider a stock (S_0=??) and call option (K=??)
    - compute payoff for option at each terminal node, then work backward.
  - Step 2: Construct *RHP* of stock and call that *replicates* a risk-free investment.
    - *Hedge ratio* for *RHP* at each node is $\Delta = (f_u - f_d) / (S_0u - S_od)$.
  - Step 3: Price call at each node by setting *return on RHP equal to riskless rate*.
    - $\Delta S_0 - f = (\Delta S_0u - f_u)*e^{-r(\delta t)}$ OR $(\Delta S_0d - f_d)*e^{-r(\delta t)}$

- **Risk Neutral Valuation** - Option pricing by arbitrage produces a pricing method equivalent to risk-neutrality. Simplifying gives (eliminate Δ):
  - $f = (pf_u + (1-p) f_d)*e^{-r(\delta t)}$ where $p = (e^{r(\delta t)} - d)/(u-d)$.

- Risk neutrality interpretation: Cost = E(Payoff discounted at risk-free rate).
  - Using this p implicitly implies $S_0 = E(S_T)*e^{-r(\delta t)}$.
  - These p's are not "real" -- actual probabilities discounted at risk adj rate.
  - However, for relative option pricing, assuming risk-neutrality is OK!
  - Choose parameters $(u, d, \delta t)$ carefully (usually $d = 1/u$).
  - Note that $0 \le p \le 1$ as long as $(1/u) \le e^{r(\delta t)} \le u$.

9 Binomial Model with Two Steps



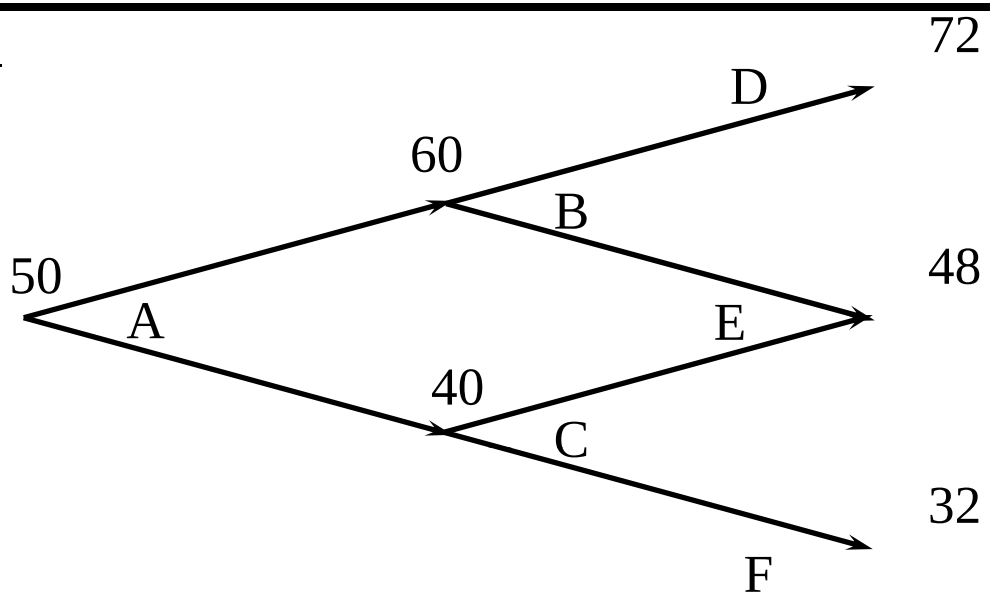
European call K=21;
T=6-mo. r=12%

- Binomial model with two-steps – start at end nodes and work backwards.
  - Use the arbitrage process previously described
  - Use risk-neutral pricing formulas.

- $p = (e^{r(\delta t)} - d) / (u - d) = (e^{0.12\times0.25} - 0.9)/(1.1 - 0.9) = 0.65$ and $(1-p) = 0.35$.

- $f = (pf_u + (1-p) f_d) * e^{-r(\delta t)}$
  f at node B = $(0.65\times3.2 + 0.35\times0) * e^{-0.12\times0.25} = 2.03$
  f at node C = 0
  f at node A = $(0.65\times2.03 + 0.35\times0) * e^{-0.12\times0.25} = 1.28$

10 Binomial Model – European Put



European Put K=52;
T=2 yr. r=5%

- Binomial model with two-steps on European Put Option –
  - Use (1) binomial model or use (2) risk-neutral pricing formulas.

- $p = (e^{r\delta t} - d) / (u - d) =$

- $f = (pf_u + (1-p) f_d) * e^{-r\delta t}$
  f at node B =
  f at node C =
  f at node A =

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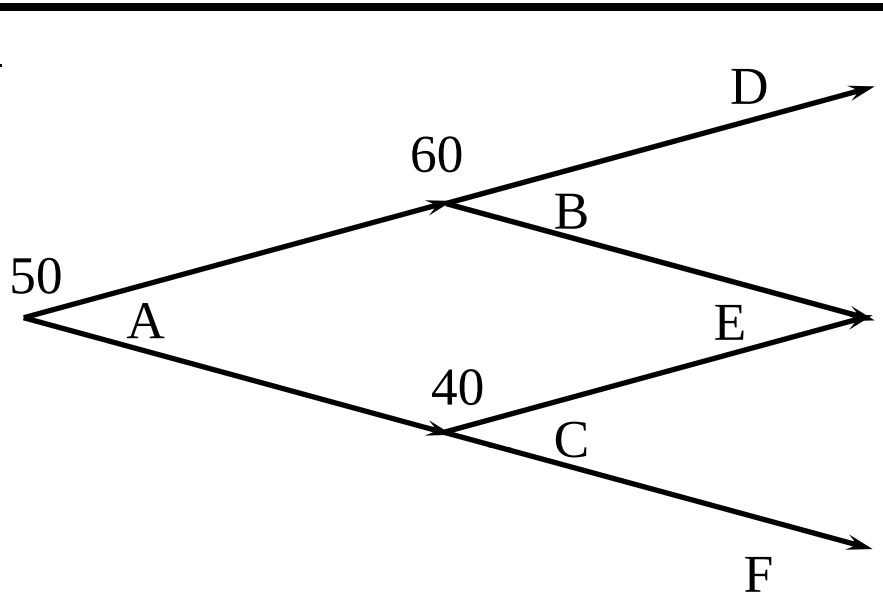
Binomial Model – American Put



American Put K=52;
T=2 yr. r=5%

Value of Early Exercise
= 5.09-4.19=0.80

- Binomial Model with two steps on American Put Option.
  - End notes have same value. Earlier nodes use greater of:
    (1) value given by method; (2) payoff from early exercise.

- $p = (e^{r\delta t} - d) / (u - d) = (e^{0.05\times 1} - 0.8) / (1.2 - 0.8) = 0.63$ and $(1-p) = 0.37$.

- $f = (pf_u + (1-p)\, f_d) * e^{-r\delta t}$
  f at node B $= (0.63\times 0 + 0.37\times 4) * e^{-0.05\times 1} = 1.41$ vs
  f at node C $= (0.63\times 4 + 0.37\times 20) * e^{-0.05\times 1} = 9.46$ vs
  f at node A =

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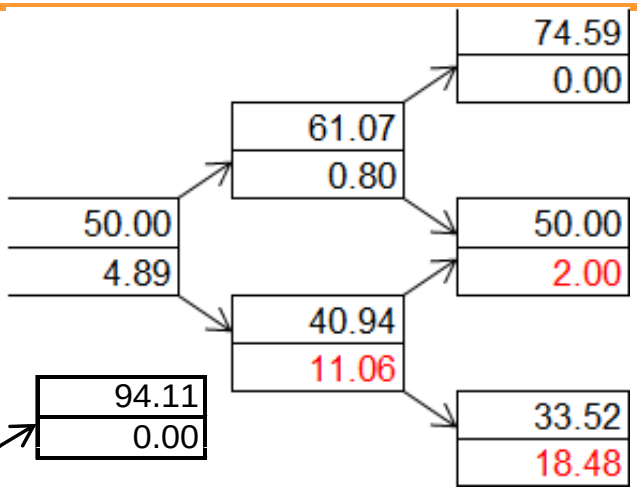
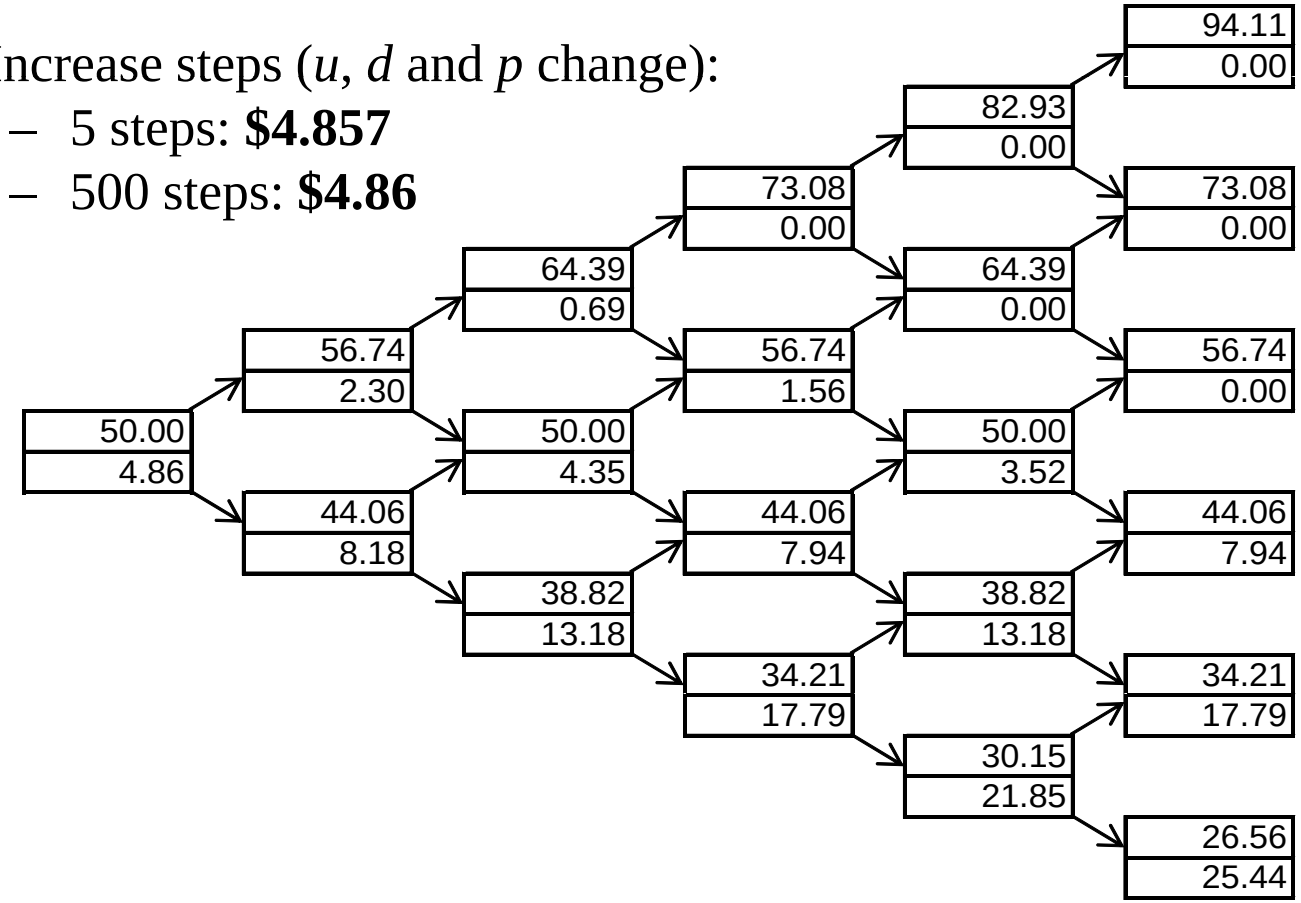
Binomial Model in Derivagem

- Previous example in *Derivagem* American Put
  - S_0=50; K=52; r=5%; σ=20%; T=2-yr; steps=2.
    $u = e^{\sigma(\delta t)^{1/2}}$=1.21; d=1/u=0.82
    p=($e^{r\,\delta t}$–d)/(u–d)=0.58; δt=1-yr
    Put = $4.89



- Increase steps (u, d and p change):
  - 5 steps: **$4.857**
  - 500 steps: **$4.86**



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Binomial Model: Comments

- American options with binomial model
  - Just compare/Insert implied option price versus exercise price at each node.
  - Delta changes through tree, so must have opportunity to rebalance.

- Binomial model used by practitioners – Note: $0 \le p \le 1$ iff $(1/u) \le e^{r(\delta t)} \le u$.
  - Use 30+steps (2^{30}=1B paths) with jumps determined by stock volatility.
  - Set $u = e^{\sigma(\delta t)^{1/2}}$; $d = 1/u = e^{-\sigma(\delta t)^{1/2}}$; $p = (e^{r\delta t} - d)/(u - d)$; or $a = e^{r\delta t}$ growth rate.

- Dividends
  - Continuous dividend yield (q) (stock index) – set $a = e^{(r-q)\delta t}$.
    - For options on stock index, FK (q=foreign risk-free rate), futures ($q=r$)
  - Dollar dividend (indiv stock) – Draw tree for stock price less PV of dividends
    - Create new tree by adding PV of div at each node. Tree recombines!

- Is there a simple formula to price options?
  - Assume price process is continuous r.w. (distribution becomes lognormal).
  - Binomial model then "second order differential equation" (Brownian motion)
  - Solution produces "Black-Scholes" formula for European options.

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Binomial Model: More Comments

- Methods for valuing options:

- Binominal model
  - **Easy:** Use "risk neutral" prob and discount at riskless rate.
  - Hard: Use "risk adjusted" prob and discount at risk adjusted required rate.

- Simulation – carefully chosen *u* and *d* imply stock returns (c.c.) normally dist.
  - Simulate many stock price paths based on c.c. returns with normal dist. Find option price for each possible stock price path. Average all outcomes.
    - **Easy:** Use riskless return and discount at riskless rate.
    - Hard: Use risk adjusted stock return and discount at risk adjusted rate.
  - Disadvantage: requires many simulations for good estimate of option value.
  - Advantage: works for "path dependent options"

- Black-Scholes formula – essentially binomial model with small "delta t"
  - Advantage: fast and easy
  - Disadvantage: only good for Euro options (or options not exercised early)